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BOUSSINESQ'S PROBLEM FOR A RIGID CONE

By A. E. H. LOVE (*Oxford*)

[Received 1 March 1939]

1. In 'Boussinesq's Problem' an elastic solid body, which is regarded as being bounded by a plane and otherwise unlimited, is deformed by normal pressure applied to part of the plane boundary, the rest of the plane being free. In one form of the problem the pressed area and the distribution of pressure over it are given, and it is desired to determine the normal displacement of a point of the initially plane boundary; in the other form the normal displacement of a point within the pressed area is given, and the distribution of pressure over that area is to be determined. In this second form of the problem the elastic solid is considered to be deformed by there being pressed against it a perfectly rigid body of given shape. The case in which the surface of this rigid body is a surface of revolution, with its axis at right angles to the initially plane boundary and finite curvature at its vertex, and the case in which it is a right circular cylinder, with its axis at right angles to the initially plane boundary and a plane end at right angles to that axis, were both solved by Boussinesq (**1**). The former case includes that where the rigid body is spherical, the latter may be referred to as that of the 'cylindrical die'.

These solutions have been found to be of practical importance (**2**), and it is desired to obtain also the solution of the problem in the case where the surface of the rigid body is a right circular cone with its axis at right angles to the plane, and its vertex penetrating into the region initially occupied by the solid.

2. The initially plane boundary of the elastic solid will be taken to be the plane $z = 0$, and the positive sense of the axis of z will be taken to be directed towards the interior of the solid. The axis of the cone will be taken to lie along the axis of z. Use will be made of a system of cylindrical coordinates ρ, ϕ, z, the distance of a point from the axis of z being denoted by ρ. The displacement of any point of the solid in the direction of the axis of z will be denoted by w. The value of w for the point of the solid that is at the vertex of the cone will be denoted by w_0. If 2α is the vertical angle of the cone, the equation to its surface is

$$z = w_0 - \rho \cot \alpha.$$

The strained surface of the solid will fit the cone over the part between the vertex and a certain circular section. The radius of this circle will be denoted by c. Then the pressed area is given by

$$z = 0, \qquad \rho \leqslant c,$$

and the value of w on the pressed area is given by

$$w = w_0 - \rho \cot \alpha \quad (0 \leqslant \rho \leqslant c).$$

The pressure applied to the pressed area will be denoted by p.

3. From the general theory (**3**) it is known that in axially symmetrical cases the displacement and stress at any point can be expressed in terms of a function V which satisfies the conditions:

(i) V is harmonic everywhere except on the pressed area;

(ii) on the pressed area $V = Aw$, where A is a certain constant;

(iii) on the plane $z = 0$ outside the pressed area $\partial V/\partial z = 0$.

Condition (i) implies that $\nabla^2 V = 0$ everywhere except on the pressed area, that V and its derivatives of the first and second orders with respect to ρ and z are continuous in the same region, and that $V \to 0$ at infinite distances in the order of the reciprocal of the distance or a higher order. In condition (ii) the constant A is given by

$$A = \frac{\pi E}{1-\sigma^2},$$

where E and σ are the Young's modulus and Poisson's ratio of the material of the solid. These constants E and σ will be used in this paper instead of Lamé's constants λ and μ which were used in the paper (**3**). We recall the relations

$$E = 2\mu(1+\sigma), \qquad \lambda(1-2\sigma) = 2\mu\sigma.$$

The pressure p is connected with V by the equation

$$p = -\frac{1}{2\pi} \lim_{z \to +0} \left(\frac{\partial V}{\partial z}\right),$$

the notation indicating that the limit is approached by allowing z to decrease to zero. It is seen that V is the Newtonian potential due to fictitious matter distributed over the pressed area with surface-density equal to p. As V is determined by the conditions (i), (ii), (iii), p also is determined by these conditions.

4. The most obvious way of finding V, when the boundary of the pressed area is a circle, and the value of w on it is a function of ρ, would seem to be to seek a series of zonal spheroidal harmonics which

shall be equal to this function when $z = 0$ and $\rho < c$, and have a zero z-derivative when $z = 0$ and $\rho > c$. The zonal spheroidal harmonics will be functions of elliptic coordinates η and θ, which are most conveniently introduced by the equation

$$z+i\rho = c\sinh(\eta+i\theta).$$

The variables η, θ, ϕ are a set of orthogonal curvilinear coordinates, and the ranges of variation of them are similar to those of spherical polar coordinates, viz. η is 0 on ($z = 0$, $\rho \leqslant c$), elsewhere $\eta > 0$, and $\eta \to \infty$ when ρ or $z \to \infty$; θ is 0 on ($\rho = 0$, $z > 0$), and π on ($\rho = 0$, $z < 0$), and elsewhere $0 < \theta < \pi$; in particular $\theta = \frac{1}{2}\pi$ on $z = 0$, $\rho \geqslant c$; ϕ can be considered as lying between $-\pi$ and π. It is strictly correct to regard a locus $\phi =$ constant as an infinite half-plane bounded by the axis of z.

From the theory given by Hobson (**4**) it appears that V must be of the form

$$V = \sum_n A_n \frac{Q_n(i\lambda)}{Q_n(i0)} P_n(\mu),$$

where $\qquad \lambda = \sinh\eta, \qquad \mu = \cos\theta,$

P_n is the symbol for Legendre's functions of the first kind, Q_n is the symbol for those of the second kind, and A_n is a constant such that $\sum A_n P_n(\mu)$ is the Legendre's series for Aw on $z = 0$, $\rho < c$. To avoid ambiguity the functions Q_n will be understood to be defined as by Hobson (**5**).

5. In the cone problem, w on the pressed area is given by

$$w = w_0 - c\cot\alpha\sin\theta,$$

and the Legendre's series for $\sin\theta$ is known (**6**) to be given by

$$\sin\theta = \tfrac{1}{2}\pi\left[\frac{1}{2}-\frac{5}{4}\frac{1}{2^2}P_2(\mu)-\sum_{m=2}^{\infty}(4m+1)\frac{2m-1}{2m+2}\left\{\frac{1.3\ldots(2m-3)}{2.4\ldots 2m}\right\}^2 P_{2m}(\mu)\right].$$

Hence we get

$$\left.\begin{aligned} &A_0 = A(w_0-\tfrac{1}{4}\pi c\cot\alpha), \qquad A_2 = A\frac{\pi}{2}\frac{5}{4}\frac{1}{2^2}c\cot\alpha \\ &A_{2m} = A\tfrac{1}{2}\pi(4m+1)\frac{2m-1}{2m+2}\left\{\frac{1.3\ldots(2m-3)}{2.4\ldots 2m}\right\}^2 c\cot\alpha \quad (m \geqslant 2) \\ &A_{2m-1} = 0 \quad (m \geqslant 1) \end{aligned}\right\}.$$

In the region $z = 0$, $\rho > c$ we have

$$\frac{\partial V}{\partial z} = \left\{h\left(-\frac{\partial V}{\partial \theta}\right)\right\}_{z=0},$$

where $h = |d(\eta+i\theta)/d(z+i\rho)|$ and $\theta = \frac{1}{2}\pi$,

and this is seen to be the same as

$$\{1/(c\sinh\eta)\}(\partial V/\partial\mu)_{\mu=0},$$

which vanishes because $dP_{2m}(\mu)/d\mu$ vanishes with μ for all values of m.

The function V has now been determined.

6. To find the corresponding formula for p we observe that, in the region $z = 0$, $\rho < c$,

$$\lim_{z\to+0}\left(\frac{\partial V}{\partial z}\right) = \left(h\frac{\partial V}{\partial\eta}\right)_{\eta=0},$$

in which $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The right-hand member is the same as

$$\{1/(c\cos\theta)\}(\partial V/\partial\lambda)_{\lambda=0}.$$

Hence we get

$$p = -\frac{1}{2\pi c\mu}\sum_{n=0}^{\infty} A_n\, i\frac{Q_n'(i0)}{Q_n(i0)}P_n(\mu) \quad (0 \leqslant \mu \leqslant 1),$$

in which $Q_n'(\zeta)$ means $\{dQ_n(\zeta)/d\zeta\}$, and is to be evaluated for $\zeta = i\lambda$ and $\lambda = 0$.

According to Hobson (7) the Q_n functions, as defined by him, satisfy the same recurrence equations as the P_n functions. From his definitions it is easy to obtain the particular formulae

$$Q_0(i\lambda) = -i\cot^{-1}\lambda, \qquad Q_1(i\lambda) = \lambda\cot^{-1}\lambda - 1,$$

and from these it is easy, by aid of the recurrence equations, to prove that

$$Q_0(i0) = -\tfrac{1}{2}i\pi, \qquad Q_{2m}(i0) = (-1)^{m+1}\frac{1.3\ldots(2m-1)}{2.4\ldots 2m}\tfrac{1}{2}i\pi,$$

$$Q_0'(i0) = 1, \qquad Q_{2m}'(i0) = (-1)^m\frac{2.4\ldots 2m}{1.3\ldots(2m-1)}.$$

The formula for p now becomes

$$p = \frac{A}{2\pi c\mu}\left[\frac{2w_0}{\pi} + c\cot\alpha\left\{-\frac{1}{2} + \sum_{m=1}^{\infty}\frac{4m+1}{(2m-1)(2m+2)}P_{2m}(\mu)\right\}\right],$$

or, as it may be written,

$$p = \frac{A}{2\pi c\mu}\left[\left(\frac{2w_0}{\pi} - c\cot\alpha\right) + c\cot\alpha\left\{\tfrac{1}{2}P_0 + (P_2 + \tfrac{1}{4}P_2) + (\tfrac{1}{3}P_4 + \tfrac{1}{6}P_4) + \ldots + \right.\right.$$

$$\left.\left. + \left(\frac{1}{2m-1}P_{2m} + \frac{1}{2m+2}P_{2m}\right) + \ldots\right\}\right].$$

In the last expression the terms in the bracket multiplying $c\cot\alpha$ can be regrouped, without altering their order, as

$$\frac{1}{1.2}(P_0+2P_2)+\frac{1}{3.4}(3P_2+4P_4)+\ldots+$$

$$+\frac{1}{(2m-1)2m}\{(2m-1)P_{2m-2}+2mP_{2m}\}+\ldots,$$

which, by the recurrence equation

$$nP_n+(n-1)P_{n-2} = (2n-1)\mu P_{n-1},$$

can be expressed as

$$\mu\left\{\frac{1}{1.2}3P_1+\frac{1}{3.4}7P_3+\ldots+\frac{1}{(2m-1)(2m)}(4m-1)P_{2m-1}+\ldots\right\}.$$

Hence we get

$$p = \frac{A}{2\pi c}\left\{\left(\frac{2w_0}{\pi}-c\cot\alpha\right)\frac{1}{\mu}+c\cot\alpha\sum_{m=1}^{\infty}\frac{4m-1}{(2m-1)2m}P_{2m-1}(\mu)\right\}.$$

This expression shows that, unless

$$w_0 = \tfrac{1}{2}\pi c\cot\alpha,$$

p will become infinite at the circular boundary of the pressed area. As the pressure will certainly not become infinite at this circle, it will henceforth be assumed that c and w_0 are connected by this equation.

The expression for p also shows that $p\to\infty$ when $\rho = 0$, for the series diverges when $\mu = 1$. This result indicates great concentration of stress near the vertex of the cone, which is to be expected. It may be remembered that in Boussinesq's solution for the case of the cylindrical die there is infinite pressure at the edge of the die. The conclusion to be drawn would seem to be that the pressure against a solid of a very rigid body, with a projecting very sharp edge or point, is likely to produce some permanent set in the part of the solid that is near to the edge or point.

7. The formula for p is now

$$p = \frac{A}{2\pi}\cot\alpha\sum_{m=1}^{\infty}\frac{4m-1}{(2m-1)2m}P_{2m-1}(\mu).$$

The series on the right is the Legendre's series for $\frac{1}{2}\log\{(1+\mu)/(1-\mu)\}$, for, in the first place, this function satisfies the conditions which are

sufficient to secure that it can be represented by such a series (**8**); and, in the second place, if the series is

$$\sum a_n P_n(\mu),$$

n is clearly uneven, and the coefficient a_n, for uneven n, is given by

$$\frac{2}{2n+1}a_n = \int_{-1}^{1} \frac{1}{2}\log\frac{1+\mu}{1-\mu}P_n(\mu)\,d\mu,$$

where the value of the integral can be proved to be $2/\{n(n+1)\}$.

We have, in fact,

$$\int_{-1}^{1} \frac{1}{2}\log\frac{1+\mu}{1-\mu}P_n(\mu)\,d\mu = -\int_{-1}^{1} \frac{1}{2}\log\frac{1+\mu}{1-\mu}\frac{d}{d\mu}\left\{\frac{1-\mu^2}{n(n+1)}\frac{dP_n}{d\mu}\right\}d\mu$$

$$= \int_{-1}^{1} \frac{1-\mu^2}{n(n+1)}\frac{1}{2}\left(\frac{1}{1+\mu}+\frac{1}{1-\mu}\right)\frac{dP_n}{d\mu}\,d\mu = \frac{2}{n(n+1)} \quad (n \text{ uneven}).$$

Hence we find $$p = \frac{A}{4\pi}\cot\alpha\log\frac{1+\mu}{1-\mu}.$$

8. The result that p can be expressed in a closed form suggests the possibility that V also might be expressed in a form free from infinite series. The natural way of finding V, as the potential due to surface-density p, would seem to be to evaluate by direct integration the potential, say V_a, at a point on the axis of symmetry, and then seek to generalize the expression so found (**9**).

If we write $$V_a = (A/4\pi)\cot\alpha V_0,$$

we shall have $$V_0 = \int_0^c \log\frac{1+\mu}{1-\mu}\frac{1}{\sqrt{(\rho^2+z^2)}}2\pi\rho\,d\rho,$$

in which $\mu = \sqrt{(1-\rho^2/c^2)}$ and $0 \leqslant \mu \leqslant 1$.

It will be sufficient to consider the case where z is positive. Then it is convenient to write

$$\log\frac{1+\mu}{1-\mu} = \log\frac{(1+\mu)^2}{1-\mu^2} = 2\left\{\log(1+\mu)-\log\frac{\rho}{c}\right\},$$

and we have

$$\frac{V_0}{4\pi} = \int_0^c \log(1+\mu)\frac{d}{d\rho}\{\sqrt{(\rho^2+z^2)}-z\}\,d\rho - \int_0^c \log\frac{\rho}{c}\frac{d}{d\rho}\{\sqrt{(\rho^2+z^2)}-z\}\,d\rho.$$

Both the integrals in this can be integrated by parts, and in each case the terms at the limits vanish, so we get

$$\frac{V_0}{4\pi} = -\int_0^c \frac{\sqrt{(\rho^2+z^2)}-z}{1+\mu}\frac{d\mu}{d\rho}\,d\rho + \int_0^c \frac{\sqrt{(\rho^2+z^2)}-z}{\rho}\,d\rho.$$

As regards the first of these we have

$$-\int_0^c \frac{\sqrt{(\rho^2+z^2)}-z}{1+\mu}\frac{d\mu}{d\rho}\,d\rho = \int_0^1 \frac{\sqrt{(z^2+c^2-c^2\mu^2)}-z}{1+\mu}\,d\mu.$$

This can be evaluated by ordinary methods, and its value is found to be

$$-z\log 2+z\log\left\{1+\frac{z}{\sqrt{(z^2+c^2)}}\right\}+c\tan^{-1}\frac{c}{z}+z-\sqrt{(z^2+c^2)}.$$

We also have, again by ordinary methods,

$$\int_0^c \frac{\sqrt{(\rho^2+z^2)}-z}{\rho}\,d\rho = -z\log\frac{\sqrt{(z^2+c^2)}+z}{z}+\sqrt{(z^2+c^2)}+z\log 2-z.$$

Hence we find, when $z > 0$,

$$V_a = A\cot\alpha\left\{z\log\frac{z}{\sqrt{(z^2+c^2)}}+c\tan^{-1}\frac{c}{z}\right\}.$$

9. The key to the method of generalizing this formula so as to obtain an expression for V which shall be valid at all points in the region $z > 0$, whether they are on the axis of z or not, is found in the fact that, if r denotes distance from the origin, the function

$$z\log(z+r)-r,$$

sometimes called the 'second logarithmic potential', is a solution of Laplace's equation $\nabla^2 = 0$. At points on the half-line given by $\rho = 0$ and $z > 0$ this function becomes

$$z(\log z+\log 2-1),$$

so we may generalize the term $z\log z$ which occurs in V_a as

$$z\log(z+r)-r+z(1-\log 2).$$

It occurred to me to try adding together three second logarithmic potentials, one of them the above function related to the origin, and the other two related in the same way to the points

$$\rho = 0, \qquad z = \pm ic.$$

Let R_1, R_2 denote the 'complex distances' of a point (x,y,z) from these points, the phases (or arguments) of R_1 and R_2 being adjusted so that, on the half-line ($\rho = 0$, $z > 0$), R_1 and R_2 may be equal respectively to $z-ic$ and $z+ic$. If we write, as a definition,

$$W = \{z\log(z+r)-r\}-\tfrac{1}{2}\{(z-ic)\log(z-ic+R_1)-R_1\}-$$
$$-\tfrac{1}{2}\{(z+ic)\log(z+ic+R_2)-R_2\},$$

the value of W on the half-line will be

$$z[\log 2z-\tfrac{1}{2}\log\{2(z-ic)2(z+ic)\}]+\tfrac{1}{2}ic\log\frac{z-ic}{z+ic},$$

or
$$z\log\frac{z}{\sqrt{(z^2+c^2)}}+c\tan^{-1}\frac{c}{z}.$$

So that we have, on the half-line,

$$V = AW\cot\alpha.$$

10. We are going to show that the function $AW\cot\alpha$ satisfies the conditions laid down for V in § 3.

We begin by explaining how the moduli of R_1, R_2 are determined and how the phases of R_1 and R_2 are chosen. So far we have the definitions

$$R_1^2 = \rho^2+(z-ic)^2, \qquad R_2^2 = \rho^2+(z+ic)^2,$$

from which we see that, if we take $R_2^2 = |R_2|^2e^{iv}$, we can have $R_1^2 = |R_1|^2e^{-iv}$. With this notation

$$|R_1|^2\cos v = \rho^2+z^2-c^2 = |R_2|^2\cos v$$

and
$$|R_1|^2\sin v = 2zc = |R_2|^2\sin v.$$

Also $|R_1|^2 = |R_2|^2 = r_1 r_2$, where r_1, r_2 denote the respective distances of the point (x,y,z) from the points S, H which are on the focal circle, $z = 0$, $\rho = c$, S being in the axial half-plane through (x,y,z), and H being in the other half of the same axial plane. For we have

$$|R_1|^4 = (\rho^2+z^2-c^2)^2+4z^2c^2 = \{(\rho-c)^2+z^2\}\{(\rho+c)^2+z^2\} = r_1^2 r_2^2.$$

In the region $z > 0$, with which alone we are concerned, $\sin v$ is positive and $\cos v$ is positive or negative according as $r > c$ or $r < c$, so that there is always a value of v between 0 and π. We choose this value to express the phase of R_2^2. It is determined without ambiguity by the condition of lying between 0 and π, together with the equation

$$\tan v = \frac{2cz}{\rho^2+z^2-c^2}.$$

This equation can be written

$$\rho^2+(z-c\cot v)^2 = c^2\operatorname{cosec}^2 v,$$

and it shows that the point (x, y, z) lies on a circle, whose centre is at $\rho = 0$, $z = c\cot v$, and whose radius is $c\operatorname{cosec} v$. The coordinate z being positive, the angle v is the angle subtended at (x, y, z) by the line SH.

We choose the phases of R_1 and R_2 to be $-\frac{1}{2}v$ and $\frac{1}{2}v$. Then the values of R_1 and R_2 on the half-line $\rho = 0$, $z > 0$ will be $z-ic$ and $z+ic$. In general we shall have

$$R_1 = \sqrt{(r_1 r_2)}(\cos\tfrac{1}{2}v - i\sin\tfrac{1}{2}v), \qquad R_2 = \sqrt{(r_1 r_2)}(\cos\tfrac{1}{2}v + i\sin\tfrac{1}{2}v),$$

so that

$$R_1+R_2 = 2\sqrt{(r_1 r_2)}\cos\tfrac{1}{2}v, \qquad R_2-R_1 = 2i\sqrt{(r_1 r_2)}\sin\tfrac{1}{2}v.$$

These can be expressed in terms of the elliptic coordinates η, θ. For

$$\{2\sqrt{(r_1 r_2)}\cos\tfrac{1}{2}v\}^2$$
$$= 2r_1 r_2(1+\cos v) = 2r_1 r_2+(r_1^2+r_2^2-4c^2) = (r_1+r_2)^2-4c^2,$$

and

$$\{2\sqrt{(r_1 r_2)}\sin\tfrac{1}{2}v\}^2$$
$$= 2r_1 r_2(1-\cos v) = 2r_1 r_2-(r_1^2+r_2^2-4c^2) = 4c^2-(r_2-r_1)^2.$$

Also $$r_1+r_2 = 2c\cosh\eta, \qquad r_2-r_1 = 2c\sin\theta.$$

Hence we have

$$R_1+R_2 = 2c\sinh\eta, \qquad R_2-R_1 = 2ic\cos\theta,$$

the signs of R_1+R_2 and $-i(R_2-R_1)$ being positive.

The product $R_1 R_2$ is $c^2(\cosh^2\eta-\sin^2\theta)$ or $c^2(\sinh^2\eta+\cos^2\theta)$.

11. We can now express W in a real form. We have

$$W = z\log(z+r)-\tfrac{1}{2}z\log\{(z-ic+R_1)(z+ic+R_2)\}+$$
$$+\tfrac{1}{2}ic\log\frac{z-ic+R_1}{z+ic+R_2}-r+\tfrac{1}{2}(R_1+R_2).$$

Also

$$(z-ic+R_1)(z+ic+R_2) = z^2+z(R_1+R_2)+R_1 R_2+ic(R_1-R_2)+c^2$$
$$= c^2(\sinh^2\eta\cos^2\theta+2\sinh^2\eta\cos\theta+\sinh^2\eta+\cos^2\theta+2\cos\theta+1)$$
$$= c^2\cosh^2\eta(1+\cos\theta)^2.$$

Further, $$\log\frac{z-ic+R_1}{z+ic+R_2} = \log\frac{(\sinh\eta-i)(1+\cos\theta)}{(\sinh\eta+i)(1+\cos\theta)}$$
$$= -2i\cot^{-1}\sinh\eta,$$

the principal value of the logarithm, and the value of the inverse cotangent that is between 0 and $\frac{1}{2}\pi$, being chosen. Hence we get

$$W = z\log\frac{z+r}{c\cosh\eta(1+\cos\theta)}+c\cot^{-1}\sinh\eta-r+c\sinh\eta.$$

On the half-line $\rho = 0$, $z > 0$ this reduces to the value found before for $(1/A\cot\alpha)V_a$ for on this half-line $r = z = c\sinh\eta$ and $\cos\theta = 1$.

On the plane $z = 0$ in the region $\rho < c$, it reduces to

$$\tfrac{1}{2}\pi c-\rho,$$

for, in this region, $z = 0$, $\eta = 0$, $r = \rho$.

On the plane $z = 0$ in the region $\rho > c$, it reduces to

$$c\sin^{-1}(c/\rho)+\sqrt{(\rho^2-c^2)}-\rho.$$

To approximate to W when $r \to \infty$ and $z > 0$, we note that

$$r = c\sqrt{(\cosh^2\eta-\cos^2\theta)},$$

so that

$$z+r = c\cosh\eta\left\{\left(1-\frac{1}{\cosh^2\eta}\right)^{\frac{1}{2}}\cos\theta+\left(1-\frac{\cos^2\theta}{\cosh^2\eta}\right)^{\frac{1}{2}}\right\},$$

which is approximately equal to $c\cosh\eta(1+\cos\theta)(1-\frac{1}{2}\cos\theta\operatorname{sech}^2\eta)$.

Hence $z\log\dfrac{z+r}{c\cosh\eta(1+\cos\theta)}$ is approximately

$$-\tfrac{1}{2}c\sinh\eta\cos^2\theta\operatorname{sech}^2\eta, \quad \text{i.e.} \quad -\tfrac{1}{2}(c^2/r)\cos^2\theta.$$

Also $c\cot^{-1}\sinh\eta$ is approximately $c/\sinh\eta$, i.e. c^2/r. And $-r+c\sinh\eta$ is approximately

$$-c\cosh\eta\left\{\left(1-\frac{1}{2}\frac{\cos^2\theta}{\cosh^2\eta}\right)-\left(1-\frac{1}{2}\frac{1}{\cosh^2\eta}\right)\right\},$$

or it is approximately $-\frac{1}{2}(c^2/r)\sin^2\theta$. It follows that

$$\lim_{r\to\infty}(rW)-\tfrac{1}{2}c^2 = 0.$$

12. Real forms for the derivatives of W with respect to z and ρ are obtained by differentiating the expression for W as a sum of second logarithmic potentials. We begin with $\partial W/\partial z$.

We have

$$\frac{\partial W}{\partial z} = \log(z+r)-\tfrac{1}{2}\log\{(z-ic+R_1)(z+ic+R_2)\}$$

$$= \log\frac{z+r}{c\cosh\eta(1+\cos\theta)}.$$

On the plane $z = 0$ in the region $\rho < c$, this reduces to

$$\log\{\sin\theta/(1+\cos\theta)\},$$

which is the same as $-\frac{1}{2}\log\{(1+\mu)/(1-\mu)\}$.

On the plane $z = 0$ in the region $\rho > c$, it is zero.

The function $AW\cot\alpha$ has now been shown to satisfy the conditions laid down for V, with the exception of the condition that the function and its derivatives should be continuous. This, however, is evident from the logarithmic form of W, for each of the three second logarithmic potentials satisfies these continuity conditions in the region with which we are concerned. We can infer that

$$V = AW\cot\alpha$$

is the solution of the cone problem.

We have noted above that the value of $\partial W/\partial z$ on the pressed area is that required for $-\partial V/\partial z$ to be equal to $2\pi p$.

The resultant pressure P on the pressed area is obtained by integrating p over the area. We have

$$P = \frac{A\cot\alpha}{4\pi}\int\limits_0^c \log\frac{1+\mu}{1-\mu}2\pi\rho\,d\rho$$

$$= \tfrac{1}{2}Ac^2\cot\alpha\int\limits_0^1 \log\frac{1+\mu}{1-\mu}\mu\,d\mu.$$

Also $$\int\limits_0^1 \log\frac{1+\mu}{1-\mu}\mu\,d\mu = -\int\limits_0^1 \log\frac{1+\mu}{1-\mu}\frac{d}{d\mu}\left(\frac{1-\mu^2}{2}\right)d\mu,$$

which is found easily to be equal to 1. Hence

$$P = \tfrac{1}{2}Ac^2\cot\alpha = \tfrac{1}{2}\pi c^2\cot\alpha\{E/(1-\sigma^2)\}.$$

This result gives c when P is known.

The value of W outside the pressed area shows that the profile of the strained bounding surface of the elastic solid is given by

$$w = \{c\sin^{-1}(c/\rho)+\sqrt{(\rho^2-c^2)}-\rho\}\cot\alpha.$$

This result, with the relations between w_0, P, and c, and the formulae for p and V, constitute the solution of Boussinesq's problem for a rigid cone within the meaning usually attached to such solutions.

13. For a complete solution it would be necessary to calculate the components of displacement and stress at any point. Formulae for this purpose are known (**3**). To apply them we must obtain real expressions for some additional derivatives of W.

From the logarithmic formula we have

$$\frac{\partial W}{\partial\rho} = \left(\frac{z}{z+r}-1\right)\frac{\rho}{r}-\frac{1}{2}\left(\frac{z-ic}{z-ic+R_1}-1\right)\frac{\rho}{R_1}-\frac{1}{2}\left(\frac{z+ic}{z+ic+R_2}-R_2\right)\frac{\rho}{R_2}$$

$$= \rho\left(-\frac{1}{z+r}+\frac{1}{2}\frac{1}{z-ic+R_1}+\frac{1}{2}\frac{1}{z+ic+R_2}\right).$$

Here $\rho^2 = r^2-z^2 = R_1^2-(z-ic)^2 = R_2^2-(z+ic)^2.$

Hence

$$\frac{\partial W}{\partial\rho} = \frac{1}{\rho}[-(r-z)+\tfrac{1}{2}\{R_1-(z-ic)+R_2-(z+ic)\}]$$

$$= -\frac{r-c\sinh\eta}{\rho}.$$

The limit of this as $\rho\to 0$ is zero, and that of $\rho^{-1}(\partial W/\partial\rho)$ is finite except at $z=0$.

Of the second derivatives of W only two need be found, viz.: $\partial^2W/\partial z^2$ and $\partial^2W/\partial\rho\partial z$, for $\partial^2W/\partial\rho^2$ is given by

$$\frac{\partial^2W}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial W}{\partial\rho}+\frac{\partial^2W}{\partial z^2} = 0.$$

From the logarithmic formula for W we found

$$\frac{\partial W}{\partial z} = \log(z+r)-\tfrac{1}{2}\log(z-ic+R_1)-\tfrac{1}{2}\log(z+ic+R_2).$$

Hence

$$\frac{\partial^2W}{\partial z^2} = \frac{1}{r}-\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = \frac{1}{r}-\frac{\sinh\eta}{c(\cosh^2\eta-\sin^2\theta)}.$$

Hence also

$$\frac{\partial^2W}{\partial\rho\partial z} = \rho\left(\frac{1}{z+r}\frac{1}{r}-\frac{1}{2}\frac{1}{z-ic+R_1}\frac{1}{R_1}-\frac{1}{2}\frac{1}{z+ic+R_2}\frac{1}{R_2}\right)$$

$$= \frac{1}{\rho}\left(\frac{r-z}{r}-\frac{1}{2}\frac{R_1-z+ic}{R_1}-\frac{1}{2}\frac{R_2-z-ic}{R_2}\right)$$

$$= \frac{1}{\rho}\left[z\left\{\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)-\frac{1}{r}\right\}-ic\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]$$

$$= \frac{1}{\rho}\left[z\left\{\frac{\sinh\eta}{c(\cosh^2\eta-\sin^2\theta)}-\frac{1}{r}\right\}+\frac{\cos\theta}{\cosh^2\eta-\cos^2\theta}\right]$$

$$= \frac{1}{\rho}\left(\frac{\cosh^2\eta\cos\theta}{\cosh^2\eta-\sin^2\theta}-\frac{z}{r}\right).$$

14. The component displacements U and w in the directions of increase of ρ and z are given by

$$U = -\frac{1+\sigma}{2\pi E}\left\{(1-2\sigma)\frac{\partial\chi}{\partial\rho}+z\frac{\partial V}{\partial\rho}\right\},$$

$$w = \frac{1+\sigma}{2\pi E}\left\{2(1-\sigma)V-z\frac{\partial V}{\partial z}\right\},$$

where $\partial\chi/\partial\rho$ is the function determined by the conditions

$$(a)\quad \frac{\partial}{\partial\rho}\left(\rho\frac{\partial\chi}{\partial\rho}\right) = -\rho\frac{\partial V}{\partial z},$$

$$(b)\quad \frac{\partial}{\partial z}\left(\rho\frac{\partial\chi}{\partial\rho}\right) = \rho\frac{\partial V}{\partial\rho},$$

$$(c)\quad \text{in the region } (z=0, \rho>c),\ \rho\frac{\partial\chi}{\partial\rho} = P.$$

From these conditions it is found that

$$\frac{\rho\dfrac{\partial\chi}{\partial\rho}-P}{A\cot\alpha} = \int\limits_{\frac{1}{2}\pi}^{\theta}\left(-\rho\frac{\partial W}{\partial z}\frac{\partial\rho}{\partial\theta}+\rho\frac{\partial W}{\partial\rho}\frac{\partial z}{\partial\theta}\right)d\theta,$$

η being constant in the integration. After some reduction the result is obtained in the form

$$\frac{\rho\dfrac{\partial\chi}{\partial\rho}-P}{Ac^2\cot\alpha} = -\frac{1}{2}\left\{\left(\frac{r}{c}-\sinh\eta\right)\sinh\eta\cos\theta+\cos\theta+\right.$$
$$\left.+\cosh^2\eta\sin^2\theta\log\frac{z+r}{c\cosh\eta(1+\cos\theta)}\right\}.$$

The stress-components that do not vanish are given by

$$\widehat{\rho\rho} = \frac{1}{2\pi}\left\{2\sigma\frac{\partial V}{\partial z}-(1-2\sigma)\frac{\partial^2\chi}{\partial\rho^2}-z\frac{\partial^2 V}{\partial\rho^2}\right\},$$

$$\widehat{\phi\phi} = \frac{1}{2\pi}\left\{2\sigma\frac{\partial V}{\partial z}-(1-2\sigma)\frac{1}{\rho}\frac{\partial\chi}{\partial\rho}-\frac{z}{\rho}\frac{\partial V}{\partial\rho}\right\},$$

$$\widehat{zz} = \frac{1}{2\pi}\left(\frac{\partial V}{\partial z}-z\frac{\partial^2 V}{\partial z^2}\right),$$

$$\widehat{\rho z} = -\frac{1}{2\pi}z\frac{\partial^2 V}{\partial\rho\partial z}.$$

The only derivative in these for which no expression has been given is $\partial^2\chi/\partial\rho^2$, and this can be found from the condition (a) above.

We have thus obtained formulae from which all the components of displacement and stress at any point can be calculated if desired.

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5. Hobson, op. cit., ch. ii.
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9. The method is an adaptation of one that goes back to Legendre, and of which an account is given in N. M. Ferrers, *Spherical Harmonics*, London, 1877, ch. iii.

SUMMARY

When a rigid cone is pressed point foremost against the surface, plane when unstrained, of an elastic solid, the solid is held strained, as if by pressure applied over a certain circular patch, the 'pressed area'. The normal displacement w of a point on this area is determined by its value w_0 at the centre and the condition of fitting the cone. The pressure p at a point of the patch is to be found from the condition that the potential V due to surface-density p on the patch is a known multiple of w on the patch. From this condition V is determined in terms of w_0 and c, the radius of the patch, as an infinite series of spheroidal harmonics. The corresponding formula for p is deduced, and the condition that p must not be infinite at the boundary of the patch leads to a relation connecting w_0 and c. The series expressing p is summed, a simple closed form being obtained. From this form for p a formula for V at any point on the prolongation of the axis of the cone into the solid is deduced. This also is a simple closed form. It is generalized so as to give a closed form for V anywhere in the solid, this form being the sum of three 'second logarithmic potentials', two of them containing imaginaries. This is then expressed as a real closed form, and it is verified that

this form actually does represent the potential due to surface-density p on the patch. By means of the closed form of p the resultant pressure P between the cone and the solid is expressed in terms of c, so that c is known when P is known. By means of the closed form of V the shape of the part of the deformed surface that is outside the pressed area is obtained. To complete the solution, formulae in terms of derivatives of V, all expressed in real closed forms, are given for the components of displacement and stress at any point in the solid, so that these components can be computed if desired.

TRANSFORMATION OF HYPERGEOMETRIC INTEGRALS BY MEANS OF FRACTIONAL INTEGRATION BY PARTS

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1. THE transformation into each other of Euler's hypergeometric integrals representing $F(\alpha,\beta;\gamma;z)$, viz.

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int\limits_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-xz)^{-\alpha}\,dx \tag{1}$$

and
$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int\limits_0^1 x^{\alpha-1}(1-x)^{\gamma-\alpha-1}(1-xz)^{-\beta}\,dx, \tag{2}$$

has been the subject of investigations by several authors.* In a recent note† I proved that this transformation can be effected by means of fractional integration by parts. Moreover, fractional integration by parts can also be used in order to obtain a more general transformation of (1), yielding the functional equation

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)}\int\limits_0^1 x^{\lambda-1}(1-x)^{\gamma-\lambda-1}F(\alpha,\beta;\lambda;xz)\,dx \tag{3}$$

$$[\Re(\gamma) > \Re(\lambda) > 0;\ z \neq 1;\ |\arg(1-z)| < \pi]$$

of the hypergeometric function.

In this article I propose (i) to derive (1), (2), (3) directly from the definition

$$F(\alpha,\beta;\gamma;z) = \sum \frac{(\alpha)_r(\beta)_r}{(\gamma)_r\, r!}z^r \tag{4}$$

of the hypergeometric function‡ using the conception of fractional

* B. Riemann, *Gesammelte mathematische Werke und wissenschaftlicher Nachlass* (Leipzig, 1876), 62–78; Schellenberg, *Dissertation* (Göttingen, 1892); W. Wirtinger, *Sitzungsber. Akad. Wiss. Wien*, 111 (1902), 894–900; E. G. C. Poole, *Quart. J. of Math.* (Oxford), 9 (1938), 230–3; A. Erdélyi, ibid. 8 (1937), 200–13 and 267–77.

† See above, 129–34.

‡ $(\alpha)_0 = 1, \qquad (\alpha)_n = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)...(\alpha+n-1) \quad (n = 1,2,3,...).$

The index of summation runs in all series from 0 to ∞.

derivatives, and (ii), by transforming (1), (2), (3) by means of fractional integration by parts, to obtain some functional equations and integral representations of F.

The methods used here also apply to the corresponding problems in the theory of generalized hypergeometric series.* For the sake of simplicity, however, I shall confine myself in this article in general to the ordinary hypergeometric function. Some limiting cases of the formulae obtained in this article yield integral representations for the confluent hypergeometric function.

2. We write the rule for fractional integration by parts† in the form

$$\int_a^b u\frac{d^\nu v}{d(b-x)^\nu}\,dx = \int_a^b v\frac{d^\nu u}{d(x-a)^\nu}\,dx. \tag{5}$$

The fractional derivatives occurring in this rule can be defined by integrals, if the real part of ν is negative. Thus

$$\left.\begin{aligned} \frac{d^\nu u}{d(x-a)^\nu} &= \frac{1}{\Gamma(-\nu)}\int_a^x (x-y)^{-\nu-1}u(y)\,dy \\ \frac{d^\nu v}{d(b-x)^\nu} &= \frac{1}{\Gamma(-\nu)}\int_x^b (y-x)^{-\nu-1}v(y)\,dy \end{aligned}\right\} \quad [\Re(\nu)<0]. \tag{6}$$

If u and v are expressible by means of series of the types

$$u = \sum A_r(x-a)^{\rho+r-1}, \qquad v = \sum B_s(b-x)^{\sigma+s-1}, \tag{7}$$

then the fractional derivatives are obtainable by differentiating these series term by term and using the definition

$$\frac{d^\nu w^{\mu-1}}{dw^\nu} = \frac{\Gamma(\mu)w^{\mu-\nu-1}}{\Gamma(\mu-\nu)} \tag{8}$$

for fractional derivatives, which holds for *all values* of ν excepting $\nu = \mu$. Obviously (6) and (8) are in accordance in those cases in which both definitions have a meaning.

* e.g. equations (1.5), (2.6), (3.2), (5.2) of my paper *Quart. J. of Math.* (Oxford), 8 (1937), 267–77 are obtainable by the method of § 3 of this article.

† E. R. Love and L. C. Young, *Proc. London Math. Soc.* (2) 44 (1938), 1–34.

3. Now we can derive (1) by means of fractional derivatives. Using the definition (8) we obtain

$$F(\alpha,\beta;\gamma;z) = \sum \frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!} z^r = \frac{\Gamma(\gamma)}{\Gamma(\beta)} z^{1-\gamma} \frac{d^{\beta-\gamma}}{dz^{\beta-\gamma}} \left\{ \sum \frac{(\alpha)_r}{r!} z^{\beta+r-1} \right\}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)} z^{1-\gamma} \frac{d^{\beta-\gamma}}{dz^{\beta-\gamma}} \{z^{\beta-1}(1-z)^{-\alpha}\}. \quad (9)$$

Now, using for the fractional derivative on the right of the last equation the definition (6), we at once arrive at

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)z^{1-\gamma}}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int\limits_0^z y^{\beta-1}(1-y)^{-\alpha}(z-y)^{\gamma-\beta-1}\, dy.$$

Introducing a new variable of integration x by the equation $y = zx$, the last relation turns into (1). A very similar analysis yields (2).

Instead of (9) we can transform (4) as follows:

$$F(\alpha,\beta;\gamma;z) = \sum \frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!} z^r = z^{1-\gamma} \frac{\Gamma(\gamma)}{\Gamma(\lambda)} \frac{d^{\lambda-\gamma}}{dz^{\lambda-\gamma}} \left\{ \sum \frac{(\alpha)_r(\beta)_r}{(\lambda)_r r!} z^{\lambda+r-1} \right\}$$

$$= z^{1-\gamma} \frac{\Gamma(\gamma)}{\Gamma(\lambda)} \frac{d^{\lambda-\gamma}}{dz^{\lambda-\gamma}} \{z^{\lambda-1} F(\alpha,\beta;\lambda;z)\}. \quad (10)$$

Using again definition (6) of fractional derivatives we obtain

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)z^{1-\gamma}}{\Gamma(\lambda)\Gamma(\gamma-\lambda)} \int\limits_0^z (z-y)^{\gamma-\lambda-1} y^{\lambda-1} F(\alpha,\beta;\lambda;y)\, dy.$$

This equation yields, on putting $y = zx$, the relation (3).

The formulae (9) and (10) are generalizations of well-known formulae for positive integer values of $\beta-\gamma$ and $\lambda-\gamma$ respectively.

4. Transforming (1) by means of fractional integration by parts, I shall now prove the formula

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)} \int\limits_0^1 x^{\lambda-1}(1-x)^{\gamma-\lambda-1}(1-xz)^{-\alpha'} \times$$

$$\times F(\alpha-\alpha',\beta;\lambda;xz) F\left(\alpha',\beta-\lambda;\gamma-\lambda;\frac{(1-x)z}{1-xz}\right) dx, \quad (11)$$

which is valid provided that

$$\Re(\gamma) > \Re(\lambda) > 0 \quad \text{and} \quad z \neq 1,\ |\arg(1-z)| < \pi.$$

Using the definition (8) of fractional derivatives we see that

$$(1-x)^{\gamma-\beta-1}(1-xz)^{-\alpha'}$$
$$= (1-z)^{-\alpha'}(1-x)^{\gamma-\beta-1}\left(1+\frac{1-x}{1-z}z\right)^{-\alpha'}$$
$$= (1-z)^{-\alpha'}\sum \frac{(\alpha')_r}{r!}\left(\frac{z}{z-1}\right)^r(1-x)^{\gamma-\beta+r-1}$$
$$= (1-z)^{-\alpha'}\frac{\Gamma(\gamma-\beta)}{\Gamma(\gamma-\lambda)}\frac{d^{\beta-\lambda}}{d(1-x)^{\beta-\lambda}}\left\{\sum \frac{(\alpha')_r(\gamma-\beta)_r}{(\gamma-\lambda)_r\, r!}\left(\frac{z}{z-1}\right)^r(1-x)^{\gamma-\lambda+r-1}\right\}$$
$$= (1-z)^{-\alpha'}\frac{\Gamma(\gamma-\beta)}{\Gamma(\gamma-\lambda)}\frac{d^{\beta-\lambda}}{d(1-x)^{\beta-\lambda}}\left\{(1-x)^{\gamma-\lambda-1}F\left(\alpha',\gamma-\beta;\gamma-\lambda;\frac{(1-x)z}{z-1}\right)\right\}.$$

Now* $$F(a,b;c;w) = (1-w)^{-a}F\left(a,c-b;c;\frac{w}{w-1}\right),$$

and therefore

$$(1-x)^{\gamma-\beta-1}(1-xz)^{-\alpha'}$$
$$= \frac{\Gamma(\gamma-\beta)}{\Gamma(\gamma-\lambda)}\frac{d^{\beta-\lambda}}{d(1-x)^{\beta-\lambda}}\left\{(1-x)^{\gamma-\lambda-1}(1-xz)^{-\alpha'}F\left(\alpha',\beta-\lambda;\gamma-\lambda;\frac{(1-x)z}{1-xz}\right)\right\}.$$

Using this result we can write

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-xz)^{-\alpha}\,dx$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 x^{\beta-1}(1-xz)^{\alpha'-\alpha}\{(1-x)^{\gamma-\beta-1}(1-xz)^{-\alpha'}\}\,dx$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\lambda)}\int_0^1 x^{\beta-1}(1-xz)^{\alpha'-\alpha}\times$$
$$\times\frac{d^{\beta-\lambda}}{d(1-x)^{\beta-\lambda}}\left\{(1-x)^{\gamma-\lambda-1}(1-xz)^{-\alpha'}F\left(\alpha',\beta-\lambda;\gamma-\lambda;\frac{(1-x)z}{1-xz}\right)\right\}dx.$$

Integrating fractionally by parts this becomes, by the rule (5),

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\lambda)}\int_0^1 (1-x)^{\gamma-\lambda-1}(1-xz)^{-\alpha'}\times$$
$$\times F\left(\alpha',\beta-\lambda;\gamma-\lambda;\frac{(1-x)z}{1-xz}\right)\frac{d^{\beta-\lambda}}{dx^{\beta-\lambda}}\{x^{\beta-1}(1-xz)^{\alpha'-\alpha}\}\,dx.$$

* E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge, 1920), § 14.4.

Now

$$\frac{d^{\beta-\lambda}}{dx^{\beta-\lambda}}\{x^{\beta-1}(1-xz)^{\alpha'-\alpha}\} = \frac{d^{\beta-\lambda}}{dx^{\beta-\lambda}}\left\{\sum \frac{(\alpha-\alpha')_r}{r!} x^{\beta+r-1}z^r\right\}$$

$$= \sum \frac{(\alpha-\alpha')_r}{r!} \frac{\Gamma(\beta+r)}{\Gamma(\lambda+r)} x^{\lambda+r-1}z^r$$

$$= x^{\lambda-1}\frac{\Gamma(\beta)}{\Gamma(\lambda)} F(\alpha-\alpha', \beta; \lambda; xz),$$

and therefore

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)} \int\limits_0^1 x^{\lambda-1}(1-x)^{\gamma-\lambda-1}(1-xz)^{-\alpha'} \times$$

$$\times F(\alpha-\alpha', \beta; \lambda; xz) F\left(\alpha', \beta-\lambda; \gamma-\lambda; \frac{(1-x)z}{1-xz}\right) dx.$$

Thus far the proof given in this section only holds provided that

$$\Re(\gamma) > \Re(\beta) > 0, \quad \Re(\gamma) > \Re(\lambda) > 0; \quad |z| < 1, \quad \Re(z) < \tfrac{1}{2}.$$

By the theory of analytic continuation the result is true, however, with only the restrictions given at the beginning of this section.

5. A few particular and limiting cases of (11) may be written out fully. Omitting those which yield formulae equivalent, by the transformation theory of hypergeometric series, to (1), (2), or (3), let us put in (11)

$$\lambda = \alpha, \qquad \alpha' = \beta.$$

Then (11) becomes

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int\limits_0^1 x^{\alpha-1}(1-x)^{\gamma-\alpha-1}(1-xz)^{-\beta} \times$$

$$\times F(\alpha-\beta, \beta; \alpha; xz) F\left(\beta-\alpha, \beta; \gamma-\alpha; \frac{(1-x)z}{1-xz}\right) dx \qquad (12)$$

$$[\Re(\gamma) > \Re(\alpha) > 0;\ z \neq 1;\ |\arg(1-z)| < \pi].$$

Examining the limiting case $\alpha \to \infty$ of (11), we must make use of the known relation*

$$\lim_{\alpha\to\infty} F\left(\alpha, \beta; \gamma; \frac{z}{\alpha}\right) = {}_1F_1(\beta; \gamma; z) = \sum \frac{(\beta)_r}{(\gamma)_r} \frac{z^r}{r!}. \qquad (13)$$

* *Modern Analysis*, § 16.1.

We replace z by z/α in (11) and make α tend to ∞. Obviously, we have

$$\lim_{\alpha\to\infty}\left(1-\frac{xz}{\alpha}\right)=1$$

uniformly in x ($0 \leqslant x \leqslant 1$), and therefore the limiting form of (11) runs

$${}_1F_1(\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)}\int\limits_0^1 x^{\lambda-1}(1-x)^{\gamma-\lambda-1}{}_1F_1(\beta;\lambda;xz)\,dx \quad (14)$$

$$[\Re(\gamma)>\Re(\lambda)>0].$$

This relation is a particular case of a similar formula for generalized hypergeometric series,* and also a particular case of equation (19) of this article.

Deriving (14), we performed the limiting process $\alpha\to\infty$ under the sign of integration. This is permissible by reason of the uniform convergence of this limiting process in the range of integration.

6. I go on to transform (3) by means of fractional integration by parts. To do so, we first transform the right of (3) by means of the transformation formula†

$$F(a,b;c;w)=(1-w)^{c-a-b}F(c-a,c-b;c;w)$$

of the hypergeometric series. Hence we get

$$F(\alpha,\beta;\gamma;z)$$
$$=\frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)}\int\limits_0^1 x^{\lambda-1}(1-x)^{\gamma-\lambda-1}(1-xz)^{\lambda-\alpha-\beta}F(\lambda-\alpha,\lambda-\beta;\lambda;xz)\,dx. \quad (15)$$

Now

$$\frac{x^{\lambda-1}}{\Gamma(\lambda)}F(\lambda-\alpha,\lambda-\beta;\lambda;xz)=\sum\frac{(\lambda-\alpha)_r(\lambda-\beta)_r}{r!\,\Gamma(\lambda+r)}x^{\lambda+r-1}z^r$$
$$=\frac{d^{\mu-\lambda}}{dx^{\mu-\lambda}}\left\{\sum\frac{(\lambda-\alpha)_r(\lambda-\beta)_r}{r!\,\Gamma(\mu+r)}x^{\mu+r-1}z^r\right\}$$
$$=\frac{d^{\mu-\lambda}}{dx^{\mu-\lambda}}\left\{\frac{x^{\mu-1}}{\Gamma(\mu)}F(\lambda-\alpha,\lambda-\beta;\mu;xz)\right\},$$

* A. Erdélyi, *Quart. J. of Math.* (Oxford), 8 (1937), 267–77, equation (5.2).

† *Modern Analysis*, § 14.4.

and therefore we can write, instead of (15),

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)\Gamma(\gamma-\lambda)} \int\limits_0^1 (1-x)^{\gamma-\lambda-1}(1-xz)^{\lambda-\alpha-\beta} \times$$
$$\times \frac{d^{\mu-\lambda}}{dx^{\mu-\lambda}}\{x^{\mu-1}F(\lambda-\alpha,\lambda-\beta;\mu;xz)\}\,dx$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\mu)} \int\limits_0^1 x^{\mu-1}F(\lambda-\alpha,\lambda-\beta;\mu;xz) \times$$
$$\times \frac{d^{\mu-\lambda}}{d(1-x)^{\mu-\lambda}}\left\{\frac{(1-x)^{\gamma-\lambda-1}}{\Gamma(\gamma-\lambda)}(1-xz)^{\lambda-\alpha-\beta}\right\}dx, \quad (16)$$

using the rule of fractional integration by parts.

The fractional derivative occurring in (16) we compute according to the definition (8). We have

$$\frac{(1-x)^{\gamma-\lambda-1}}{\Gamma(\gamma-\lambda)}(1-xz)^{\lambda-\alpha-\beta}$$
$$= (1-z)^{\lambda-\alpha-\beta}\frac{(1-x)^{\gamma-\lambda-1}}{\Gamma(\gamma-\lambda)}\left(1+\frac{1-x}{1-z}z\right)^{\lambda-\alpha-\beta}$$
$$= (1-z)^{\lambda-\alpha-\beta}\sum \frac{(\alpha+\beta-\lambda)_r}{r!\,\Gamma(\gamma-\lambda)}\left(\frac{z}{z-1}\right)^r(1-x)^{\gamma-\lambda+r-1},$$

and therefore

$$\frac{d^{\mu-\lambda}}{d(1-x)^{\mu-\lambda}}\left\{\frac{(1-x)^{\gamma-\lambda-1}}{\Gamma(\gamma-\lambda)}(1-xz)^{\lambda-\alpha-\beta}\right\}$$
$$= (1-z)^{\lambda-\alpha-\beta}\sum \frac{(\alpha+\beta-\lambda)_r(\gamma-\lambda)_r}{r!\,\Gamma(\gamma-\mu+r)}\left(\frac{z}{z-1}\right)^r(1-x)^{\gamma-\mu+r-1}$$
$$= (1-z)^{\lambda-\alpha-\beta}\frac{(1-x)^{\gamma-\mu-1}}{\Gamma(\gamma-\mu)}F\left(\alpha+\beta-\lambda,\gamma-\lambda;\gamma-\mu;\frac{(1-x)z}{z-1}\right).$$

But this is equal to

$$(1-xz)^{\lambda-\alpha-\beta}\frac{(1-x)^{\gamma-\mu-1}}{\Gamma(\gamma-\mu)}F\left(\alpha+\beta-\lambda,\lambda-\mu;\gamma-\mu;\frac{(1-x)z}{1-xz}\right),$$

by the transformation formula of the hypergeometric series quoted in § 4. Putting this expression into (16) we at length arrive at the formula

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)\Gamma(\gamma-\mu)} \int\limits_0^1 x^{\mu-1}(1-x)^{\gamma-\mu-1}(1-xz)^{\lambda-\alpha-\beta} \times$$
$$\times F(\lambda-\alpha,\lambda-\beta;\mu;xz)F\left(\alpha+\beta-\lambda;\lambda-\mu;\gamma-\mu;\frac{(1-x)z}{1-xz}\right)dx, \quad (17)$$

valid for

$$\Re(\gamma) > \Re(\mu) > 0; \qquad z \neq 1, \quad |\arg(1-z)| < \pi.$$

Obviously (17) is a generalization of (3).

7. Let us examine again a few particular and limiting forms of (17). There are many particular cases yielding (3) or formulae which after some transformations of hypergeometric series turn into (3). Omitting all such cases, let us put $\lambda = \beta$. Thus the first of the two hypergeometric series on the right of (17) reduces to 1, leaving only

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)\Gamma(\gamma-\mu)} \int\limits_0^1 x^{\mu-1}(1-x)^{\gamma-\mu-1}(1-xz)^{-\alpha} \times$$
$$\times F\left(\alpha, \beta-\mu; \gamma-\mu; \frac{(1-x)z}{1-xz}\right) dx, \quad (18)$$

valid under the same restrictions as (17). This is a generalization of Euler's integral (1), reducing to (1) if $\mu = \beta$. We get a similar formula from $\lambda = \alpha$.

Now let us replace z by z/β in (17) and then make β tend to ∞. Doing so we use (13) and remark that

$$\lim_{\beta\to\infty}\left(1-\frac{xz}{\beta}\right) = 1,$$

but

$$\lim_{\beta\to\infty}\left(1-\frac{xz}{\beta}\right)^{\lambda-\alpha-\beta} = e^{xz}.$$

Having regard to this, we obtain the limiting form of (17):

$${}_1F_1(\alpha;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)\Gamma(\gamma-\mu)} \int\limits_0^1 x^{\mu-1}(1-x)^{\gamma-\mu-1}e^{xz} \times$$
$$\times {}_1F_1\{\lambda-\alpha;\mu;-xz\}\,{}_1F_1\{\lambda-\mu;\gamma-\mu;(1-x)z\}\,dx.$$

Here we use Kummer's transformation formula*

$$e^{w}\,{}_1F_1(a;c;-w) = {}_1F_1(c-a;c;w),$$

which is the limiting form of the transformation formula quoted in § 4, and get the functional equation

$${}_1F_1(\alpha;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)\Gamma(\gamma-\mu)} \int\limits_0^1 x^{\mu-1}(1-x)^{\gamma-\mu-1} \times$$
$$\times {}_1F_1\{\alpha-\lambda+\mu;\mu;xz\}\,{}_1F_1\{\lambda-\mu;\gamma-\mu;(1-x)z\}\,dx, \quad (19)$$

* *Modern Analysis*, § 16.11 (I).

valid for $$\Re(\gamma) > \Re(\mu) > 0.$$

The formula (19) is another form of the 'addition formula' for Whittaker's confluent hypergeometric function.*

It is perhaps not superfluous to notice that a great number of formulae are particular cases of (19). The simplest among these is Euler's integral of the first kind,

$$\int_0^1 x^{\mu-1}(1-x)^{\gamma-\mu-1}\,dx = \frac{\Gamma(\mu)\Gamma(\gamma-\mu)}{\Gamma(\gamma)},$$

which results from (19) if we put $\alpha = 0$, $\lambda = \mu$. Among the others the best known are: Doetsch's 'Faltungs'-formulae† concerning Hermite polynomials, Tricomi's addition formula‡ of Laguerre polynomials, Kapteyn integrals§ involving Bessel functions, and some integrals|| involving parabolic cylinder functions and Bateman's k_n-functions.

8. Let us again start with (3) and remark that

$$x^{\mu-1}F(\alpha,\beta;\lambda;xz) = \sum \frac{(\alpha)_r(\beta)_r}{(\lambda)_r\,r!}x^{\mu+r-1}z^r$$
$$= \frac{d^{\nu-\mu}}{dx^{\nu-\mu}}\left\{\sum \frac{(\alpha)_r(\beta)_r\,\Gamma(\mu+r)}{(\lambda)_r\,r!\,\Gamma(\nu+r)}x^{\nu+r-1}z^r\right\}$$
$$= \frac{d^{\nu-\mu}}{dx^{\nu-\mu}}\left\{\frac{\Gamma(\mu)}{\Gamma(\nu)}x^{\nu-1}{}_3F_2(\alpha,\beta,\mu;\lambda,\nu;xz)\right\}.$$

Thus we may write down (3) in the form

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)\Gamma(\nu)}\int_0^1 x^{\lambda-\mu}(1-x)^{\gamma-\lambda-1}\times$$
$$\times\frac{d^{\nu-\mu}}{dx^{\nu-\mu}}\{x^{\nu-1}{}_3F_2(\alpha,\beta,\mu;\lambda,\nu;xz)\}\,dx.$$

* A. Erdélyi, *Math. Zeits.* 42 (1936), 125–43, equation (3.1).

† G. Doetsch, *Math. Zeits.* 32 (1930), 587–99.

‡ F. Tricomi, *Rendiconti dei Lincei* (6) 21 (1935), 332–5.

§ See, e.g., E. T. Copson, *Proc. London Math. Soc.* (2) 33 (1932), 145–53, § 4.

|| e.g. N. G. Shabde, *J. of Indian Math. Soc.* New series 3 (1938), 146–51, § 2; N. A. Shastri, ibid. 155–63, § 5.

Integrating fractionally by parts, this becomes

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)\Gamma(\nu)}\int\limits_0^1 x^{\nu-1}{}_3F_2(\alpha,\beta,\mu;\lambda,\nu;xz)\times$$

$$\times\frac{d^{\nu-\mu}}{d(1-x)^{\nu-\mu}}\{x^{\lambda-\mu}(1-x)^{\gamma-\lambda-1}\}\,dx.$$

Now

$$\frac{d^{\nu-\mu}}{d(1-x)^{\nu-\mu}}\{x^{\lambda-\mu}(1-x)^{\gamma-\lambda-1}\}$$

$$= \frac{d^{\nu-\mu}}{d(1-x)^{\nu-\mu}}\{(1-x)^{\gamma-\lambda-1}[1-(1-x)]^{\lambda-\mu}\}$$

$$= \frac{d^{\nu-\mu}}{d(1-x)^{\nu-\mu}}\left\{\sum\frac{(\mu-\lambda)_r}{r!}(1-x)^{\gamma-\lambda+r-1}\right\}$$

$$= \frac{\Gamma(\gamma-\lambda)}{\Gamma(\gamma-\lambda+\mu-\nu)}(1-x)^{\gamma-\lambda+\mu-\nu-1}F(\mu-\lambda,\gamma-\lambda;\gamma-\lambda+\mu-\nu;1-x).$$

Hence

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(\nu)\Gamma(\gamma-\lambda+\mu-\nu)}\int\limits_0^1 x^{\nu-1}(1-x)^{\gamma-\lambda+\mu-\nu-1}\times$$

$$\times F(\mu-\lambda,\gamma-\lambda;\gamma-\lambda+\mu-\nu;1-x)\,{}_3F_2(\alpha,\beta,\mu;\lambda,\nu;xz)\,dx, \quad (20)$$

and this relation is valid, by the theory of analytic continuation, provided that

$$\Re(\lambda)>0,\ \Re(\nu)>0,\ \Re(\gamma-\lambda+\mu-\nu)>0;\ z\neq 1,\ |\arg(1-z)|<\pi.$$

Transform the ordinary hypergeometric series on the right of (20) by means of the transformation formula quoted in § 4, and (20) becomes

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(\nu)\Gamma(\gamma-\lambda+\mu-\nu)}\int\limits_0^1 x^{\lambda+\nu-\gamma-1}(1-x)^{\gamma-\lambda+\mu-\nu-1}\times$$

$$\times F\left(\gamma-\lambda,\gamma-\nu;\gamma-\lambda+\mu-\nu;1-\frac{1}{x}\right){}_3F_2(\alpha,\beta,\mu;\lambda,\nu;xz)\,dx, \quad (21)$$

$$[\Re(\lambda)>0,\ \Re(\nu)>0,\ \Re(\gamma-\lambda+\mu-\nu)>0;\ z\neq 1,\ |\arg(1-z)|<\pi],$$

thus exhibiting the symmetry in λ and ν.

9. In this section I write out some particular cases of (20) and (21).

Let us put $\nu=\alpha$ in (20). This yields

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)\Gamma(\mu)}{\Gamma(\alpha)\Gamma(\lambda)\Gamma(\gamma-\alpha+\mu-\lambda)}\int\limits_0^1 x^{\alpha-1}(1-x)^{\gamma-\alpha-\lambda+\mu-1}\times$$

$$\times F(\mu-\lambda,\gamma-\lambda;\gamma-\alpha-\lambda+\mu;1-x)F(\beta,\mu;\lambda;xz)\,dx, \quad (22)$$

$[\Re(\alpha)>0,\ \Re(\lambda)>0,\ \Re(\gamma-\alpha-\lambda+\mu)>0;\ z\neq 1,\ |\arg(1-z)|<\pi]$. We get a similar formula from $\nu=\beta$.

Again, putting $\nu=\gamma$ in (21), we arrive at

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\mu)}{\Gamma(\lambda)\Gamma(\mu-\lambda)}\int\limits_0^1 x^{\lambda-1}(1-x)^{\mu-\lambda-1}{}_3F_2(\alpha,\beta,\mu;\gamma,\lambda;xz)\,dx,$$

$$[\Re(\mu)>\Re(\lambda)>0;\ z\neq 1;\ |\arg(1-z)|<\pi]. \quad (23)$$

This is a particular case of a functional equation of ${}_3F_2$.*

A last example of this kind we obtain by putting $\lambda=\alpha$ and $\nu=\beta$ in (21). Thus,

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)\Gamma(\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha-\beta+\mu)}\int\limits_0^1 x^{\alpha+\beta-\gamma-1}(1-x)^{\gamma-\alpha-\beta+\mu-1}\times$$

$$\times(1-xz)^{-\mu}F\left(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+\mu;1-\frac{1}{x}\right)dx \quad (24)$$

$[\Re(\alpha)>0;\ \Re(\beta)>0;\ \Re(\gamma-\alpha-\beta+\mu)>0;\ z\neq 1;\ |\arg(1-z)|<\pi]$. Giving suitable values to the arbitrary parameter μ in this relation we obtain Euler's hypergeometric integrals (1) and (2) as particular cases of (24). Thus, this relation is another expedient for connecting (1) and (2).†

10. In order to obtain certain limiting forms of (21) we must deal with

$$\lim_{\mu\to\infty}F\left(\gamma-\lambda,\gamma-\nu;\gamma-\lambda+\mu-\nu;1-\frac{\mu}{t}\right).$$

If at least one of the numbers $\gamma-\lambda$, $\gamma-\nu$ happens to be a negative integer (or zero), then, in the usual manner, this limit is readily found to be

$$\lim_{\mu\to\infty}F\left(\gamma-\lambda,\gamma-\nu;\gamma-\lambda+\mu-\nu;1-\frac{\mu}{t}\right)={}_2F_0\left(\gamma-\lambda,\gamma-\nu;-\frac{1}{t}\right), \quad (25)$$

both the hypergeometric series occurring in (25) being *terminating* ones.

* See the first footnote on p. 181.

† See also § 1 of the first of my articles quoted on p. 176.

If neither of the numbers $\gamma-\lambda$, $\gamma-\nu$ is a negative integer, then (25) seems to have no significance at all, the series on the right-hand side of (25) being divergent for all finite values of t. I propose, however, to show that (25) *is valid for all values of* γ, λ, ν, if only the divergent ${}_2F_0$ *is defined* by the equation

$$
{}_2F_0\left(a, b; -\frac{1}{w}\right) = \frac{\Gamma(b-a)}{\Gamma(b)} w^a {}_1F_1(a; a-b+1; w) + \\
+\frac{\Gamma(a-b)}{\Gamma(a)} w^b {}_1F_1(b; b-a+1; w). \quad (26)
$$

I desire to add to this definition some remarks:

(i) If one of the numbers a, b happens to be a negative integer, then one of the two terms on the right of (26) vanishes, the other being easily transformable into the left of (26). If, for example, we assume a to be a negative integer $-n$ (say), the second term on the right of (26) vanishes because $1/\Gamma(-n) = 0$, and the first term can be transformed thus:

$$
\frac{\Gamma(b+n)}{\Gamma(b)} w^{-n} {}_1F_1(-n; -n-b+1; w)
$$

$$
= \frac{\Gamma(b+n)}{\Gamma(b)} \sum \frac{(-n)_r z^{r-n}}{(-n-b+1)_r r!} = \sum \frac{\Gamma(b+n)\Gamma(1-b-n) n! (-)^r}{\Gamma(b)\Gamma(1-b-n+r) r! (n-r)!} z^{r-n}
$$

$$
= \sum \frac{(-n)_{n-r}(b)_{n-r}}{(n-r)!(-z)^{n-r}} = {}_2F_0\left(-n, b; -\frac{1}{z}\right).
$$

Hence (26) is in accordance with the familiar definition of ${}_2F_0$.

(ii) When neither a nor b happens to be a negative integer or zero, then the left of (26) is known to be the asymptotic expansion of the right-hand side of this equation, for large values of $|w|$.

(iii) Definition (26) holds only for non-integer values of $a-b$. For integer values the limiting form of (26) is to be taken.

(iv) The reader familiar with the theory of confluent hypergeometric functions will notice the close relationship between our definition of ${}_2F_0$ and the definition of Whittaker's function $W_{k,m}(z)$.*

Now, after these considerations, it is easy to prove that, with the definition (26), relation (25) is true for all values of γ, λ, ν. It is sufficient to prove (25) only for non-integer values of $\lambda-\nu$. The validity of this equation for integer values of $\lambda-\nu$ then follows at once, both members of (25) being continuous functions of λ. To prove (25) for non-integer values of $\lambda-\nu$ we transform the hyper-

* *Modern Analysis*, § 16.12.

geometric series on the left of (25) by means of the transformation formula*

$$F(a,b;c;w) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(-w)^{-a}F\left(a,a-c+1;a-b+1;\frac{1}{w}\right)+$$
$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(-w)^{-b}F\left(b,b-c+1;b-a+1;\frac{1}{w}\right),$$

thus arriving at

$$F\left(\gamma-\lambda,\gamma-\nu;\gamma-\lambda+\mu-\nu;1-\frac{\mu}{t}\right)$$
$$=\frac{\Gamma(\gamma-\lambda+\mu-\nu)\Gamma(\lambda-\nu)}{\Gamma(\mu-\nu)\Gamma(\gamma-\nu)}\left(\frac{t}{\mu-t}\right)^{\gamma-\lambda}F\left(\gamma-\lambda,1-\mu+\nu;1-\lambda+\nu;\frac{t}{t-\mu}\right)+$$
$$+\frac{\Gamma(\gamma-\lambda+\mu-\nu)\Gamma(\nu-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\gamma-\lambda)}\left(\frac{t}{\mu-t}\right)^{\gamma-\nu}F\left(\gamma-\nu,1-\mu+\lambda;1+\lambda-\nu;\frac{t}{t-\mu}\right). \tag{27}$$

Making μ tend to ∞ we remark that, by Stirling's formula,†

$$\lim_{\mu\to\infty}\frac{\Gamma(\gamma-\lambda+\mu-\nu)}{\Gamma(\mu-\nu)(\mu-t)^{\gamma-\lambda}}=1,$$

and, by (13),

$$\lim_{\mu\to\infty}F\left(\gamma-\lambda,1-\mu+\nu;1-\lambda+\nu;\frac{t}{t-\mu}\right)={}_1F_1(\gamma-\lambda;1-\lambda+\nu;t).$$

Similar relations hold, of course, with λ and ν interchanged.

Hence the limiting form of (27) becomes

$$\lim_{\mu\to\infty}F\left(\gamma-\lambda,\gamma-\nu;\gamma-\lambda+\mu-\nu;1-\frac{\mu}{t}\right)$$
$$=\frac{\Gamma(\lambda-\nu)}{\Gamma(\gamma-\nu)}t^{\gamma-\lambda}{}_1F_1(\gamma-\lambda;1-\lambda+\nu;t)+$$
$$+\frac{\Gamma(\nu-\lambda)}{\Gamma(\gamma-\lambda)}t^{\gamma-\nu}{}_1F_1(\gamma-\nu;1+\lambda-\nu;t)$$
$$={}_2F_0\left(\gamma-\lambda,\gamma-\nu;-\frac{1}{t}\right),$$

according to the definition (26). Hence (25) is established.

11. I conclude by pointing out some limiting forms of (21). To obtain a first example of this kind, let us replace z in (21) by z/β and make β tend to ∞. Since

$$\lim_{\beta\to\infty}{}_3F_2\left(\alpha,\beta,\mu;\lambda,\nu;\frac{xz}{\beta}\right)={}_2F_2(\alpha,\mu;\lambda,\nu;xz), \tag{28}$$

* *Modern Analysis*, § 14.51.

† Ibid., § 12.33.

we at once get

$$ {}_1F_1(\alpha;\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(\nu)\Gamma(\gamma-\lambda+\mu-\nu)} \int\limits_0^1 x^{\lambda+\nu-\gamma-1}(1-x)^{\gamma-\lambda+\mu-\nu-1} \times $$

$$ \times F\left(\gamma-\lambda, \gamma-\nu; \gamma-\lambda+\mu-\nu; 1-\frac{1}{x}\right) {}_2F_2(\alpha,\mu;\lambda,\nu;xz)\,dx, \quad (29) $$

valid for $\quad \Re(\lambda) > 0;\ \Re(\nu) > 0;\ \Re(\gamma-\lambda+\mu-\nu) > 0.$

This integral was established in a different manner by Meijer.*

We replace x in (21) by t/μ and make μ tend to ∞. The range of integration in t runs from 0 to μ; thus we obtain in the limit an *infinite integral*. We remark that, by Stirling's formula,

$$ \lim_{\mu\to\infty} \frac{\Gamma(\gamma-\lambda+\mu-\nu)}{\Gamma(\mu)\mu^{\gamma-\lambda-\nu}} = 1, $$

and that

$$ \lim_{\mu\to\infty} \left(1-\frac{t}{\mu}\right)^{\gamma-\lambda+\mu-\nu-1} = e^{-t}, $$

and recall (25) and (28). Now we can work out the limiting process, if we suppose $\Re(z) < 1$ in order to obtain a convergent integral. With this restriction the limiting form of (21) is

$$ F(\alpha,\beta;\gamma;z) $$

$$ = \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\nu)} \int\limits_0^\infty t^{\lambda+\nu-\gamma-1} e^{-t} {}_2F_0\left(\gamma-\lambda, \gamma-\nu; -\frac{1}{t}\right) {}_2F_2(\alpha,\beta;\lambda,\nu;tz)\,dt, \quad (30) $$

valid for $\quad \Re(\lambda) > 0;\ \Re(\nu) > 0;\ \Re(z) < 1.$

Some special cases of (30) are worth mentioning. Let us put $\nu = \gamma$ in (30). Then we obtain

$$ F(\alpha,\beta;\gamma;z) = \frac{1}{\Gamma(\lambda)} \int\limits_0^\infty t^{\lambda-1} e^{-t} {}_2F_2(\alpha,\beta;\gamma,\lambda;tz)\,dt \quad (31) $$

$$ [\Re(\lambda) > 0;\ \Re(z) < 1]. $$

Again, put $\lambda = \alpha$, $\nu = \beta$ in (30). These particular values yield Goldstein's integral†

$$ F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \int\limits_0^\infty t^{\alpha+\beta-\gamma-1} e^{-(1-z)t} {}_2F_0\left(\gamma-\alpha, \gamma-\beta; -\frac{1}{t}\right) dt \quad (32) $$

$$ [\Re(\alpha) > 0;\ \Re(\beta) > 0;\ \Re(z) < 1]. $$

Other particular cases of the formulae are left to the reader.

* C. S. Meijer, *Proc. Akad. Wet. Amsterdam*, 41 (1938), 1113.

† S. Goldstein, *Proc. London Math. Soc.* (2) 34 (1932), 103–25, equation (54).

THE 'EASIER' WARING PROBLEM

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1. In a former paper† Wright defined $v(k)$ as the least value of such that every integer n can be expressed in the form

$$n = x_1^k + x_2^k \dots + x_r^k - x_{r+1}^k \dots - x_s^k, \tag{1.1}$$

where r, x_1, $x_2, \dots, x_s$ are integers and $0 \leqslant r \leqslant s$.‡ He proved that $v(2) = 3$; $4 \leqslant v(3) \leqslant 5$; $8 \leqslant v(4) \leqslant 12$; and $5 \leqslant v(5) \leqslant 10$, the results for $v(4)$ and $v(5)$ being new. Our object here is to carry these investigations further and, in particular, to find upper and lower bounds for $v(k)$ for $6 \leqslant k \leqslant 20$.

We take $k \geqslant 3$ and we write $\Delta(k, m, n)$ for the least value of s such that the congruence

$$x_1^k + x_2^k \dots + x_r^k - x_{r+1}^k \dots - x_s^k \equiv n \pmod{m} \tag{1.2}$$

is soluble for some $r = r(n)$, and

$$\Delta(k, m) = \max_n \Delta(k, m, n).$$

In the paper mentioned Wright showed that the existence of an identity in x

$$\sum_{i=1}^{j} (x + a_i)^k - \sum_{i=1}^{j} (x + a_i')^k = Cx + D, \tag{1.3}$$

in which $a_1, a_2, \dots, a_j$, $a_1', a_2', \dots, a_j'$ are integers and $C \neq 0$, implies the inequality

$$v(k) \leqslant 2j + \Delta(k, C). \tag{1.4}$$

On the other hand, a necessary condition for the solubility of (1.1) is the solubility of the congruence (1.2) for all m. Hence, if we write§

$$\Delta(k) = \max_m \Delta(k, m),$$

so that $\Delta(k)$ is the least value of s such that (1.2) has a solution for all n and m and for suitable $r = r(n, m)$, we have

$$v(k) \geqslant \Delta(k). \tag{1.5}$$

† E. M. Wright, 'An easier Waring's problem': *J. of London Math. Soc.* 9 (1934), 267–72.

‡ Here r depends on n.

§ Wright, loc. cit., uses $\Delta(k)$ with a different meaning.

In § 2 we develop in some detail the theory of the congruence (1.2) and determine various upper bounds for $\Delta(k)$ and similar auxiliary functions. We apply these results to determine a formula for $\Delta(k)$ in certain cases, notably when $\Delta(k) > k$, and to calculate $\Delta(k)$ for all $k \leqslant 36$. Much of our work follows closely that of Hardy and Littlewood on $\Gamma(k)$,† though we encounter a complication which does not arise in their problem. We have adopted the notation of P.N. 8 as far as possible and we omit proofs where these differ from the corresponding work in P.N. 4 or P.N. 8 by merely trivial modifications.

In § 3 we apply our results to determine upper and lower bounds for $v(k)$ for values of k up to $k \leqslant 20$. For this purpose we require identities of the type (1.3) and we show how these may be found.

For large values of k the best known upper bound for $v(k)$ is given by Vinogradoff's well-known result‡ for $G(k)$ in the ordinary Waring's Problem combined with the trivial inequality

$$v(k) \leqslant G(k)+1.$$

2. The numbers $\Delta(k,m)$ and $\Delta(k)$

2.1. We use $a \mid b$ to denote that a is a divisor of b, and $a \nmid b$ to denote the contrary; we write (a,b) for the greatest common factor of a and b.

We write $\Gamma(k,m,n)$ for the least value of s such that (1.2) has a solution with $r = s$ and $\Lambda(k,m,n)$ for the least value of s such that (1.2) has a solution for every r $(0 \leqslant r \leqslant s)$. We write also§

$$\Gamma(k,m) = \max_n \Gamma(k,m,n), \qquad \Gamma(k) = \max_m \Gamma(k,m),$$

$$\Lambda(k,m) = \max_n \Lambda(k,m,n), \qquad \Lambda(k) = \max_m \Lambda(k,m).$$

Hardy and Littlewood (P.N. 8) gave methods to calculate $\Gamma(k,m)$ and $\Gamma(k)$, and we give corresponding methods to calculate $\Lambda(k,m)$ and

† G. H. Hardy and J. E. Littlewood, 'Some problems of "Partitio Numerorum": (IV) The singular series in Waring's problem and the value of the number $G(k)$', *Math. Zeits.* 12 (1922), 161–88; '(VIII) The number $\Gamma(k)$ in Waring's problem', *Proc. London Math. Soc.* (2), 28 (1928), 518–42. We refer to these papers as P.N. 4 and P.N. 8 respectively.

‡ See, for example, Landau, *Über einige neuere Ergebnisse der additiven Zahlentheorie* (Cambridge, 1937), Kap. 1. Vinogradoff has recently improved his result for $k \geqslant 800$ (*Trav. Inst. Math. Tbilissi*, 5 (1938), 167–80).

§ This definition of $\Gamma(k)$ is equivalent to that of Hardy and Littlewood except when $k = 4$ (cf. P.N. 8, p. 525).

$\Lambda(k)$. The work is greatly simplified by the fact that we need only consider the case when m is a power of a prime. This is a consequence of the two following lemmas.

LEMMA 1. *If* $(m_1, m_2) = 1$, *and if, for given* n, s, r, *there is a solution of* (1.2) *when* $m = m_1$ *and a solution when* $m = m_2$, *then there is a solution when* $m = m_1 m_2$.

Let $x_i = a_i$ and $x_i = b_i$ $(i = 1, 2,..., s)$ be the solutions for $m = m_1$ and $m = m_2$ respectively. Since $(m_1, m_2) = 1$, we can find two numbers d_1, d_2 such that

$$d_1 m_2 \equiv 1 \pmod{m_1}, \qquad d_2 m_1 \equiv 1 \pmod{m_2}.$$

Then $x_i = d_1 m_2 a_i + d_2 m_1 b_i$ is a solution of (1.2) for $m = m_1 m_2$.

From Lemma 1 we deduce that, if $(m_1, m_2) = 1$, then

$$\left.\begin{aligned} \Lambda(k, m_1 m_2, n) &= \max\{\Lambda(k, m_1, n), \Lambda(k, m_2, n)\} \\ \Lambda(k, m_1 m_2) &= \max\{\Lambda(k, m_1), \Lambda(k, m_2)\} \end{aligned}\right\} \tag{2.11}$$

and similar results for $\Gamma(k, m, n)$ and $\Gamma(k, m)$. Hence

$$\Lambda(k) = \max_{p,t} \Lambda(k, p^t), \qquad \Gamma(k) = \max_{p,t} \Gamma(k, p^t),$$

where p runs through all primes and t through all positive integers. Nothing similar is true in general for $\Delta(k, m, n)$ and $\Delta(k, m)$; for, in the determination of $\Delta(k, m, n)$, r is a function of m as well as of n, so that Lemma 1 does not lead to results corresponding to (2.11). We can only say that

$$\Delta(k, m_1 m_2, n) \geqslant \max\{\Delta(k, m_1, n), \Delta(k, m_2, n)\}$$

and, in fact, it is easy to find numerical examples in which equality is false.† Thus it is easily verified that

$$\Delta(6, 8, 41) = 1, \qquad \Delta(6, 9, 41) = 4, \qquad \Delta(6, 72, 41) = 5.$$

We can, however, deduce from Lemma 1 that

$$\left.\begin{aligned} \Delta(k, m_1 m_2, n) &\leqslant \max\{\Delta(k, m_1, n), \Lambda(k, m_2, n)\} \\ \text{and} \qquad \Delta(k, m_1 m_2) &\leqslant \max\{\Delta(k, m_1), \Lambda(k, m_2)\} \end{aligned}\right\}, \tag{2.12}$$

provided that $(m_1, m_2) = 1$.

We see that a knowledge of upper bounds for $\Lambda(k, m)$ is useful if

† This difficulty appears to have been overlooked by I. Chowla in his paper 'On $\Gamma(k)$ in Waring's problem and an analogous function', *Proc. Indian Acad. Sci.*, Sect. A, 5 (1937), 269–76.

we wish to calculate $\Delta(k)$, and we therefore study $\Lambda(k,m)$ as fully as $\Delta(k,m)$. It does not seem unreasonable to suppose that

$$\Lambda(k,m) = \Gamma(k,m),$$

but we are not able to prove this. We can prove, however, that $\Lambda(k,m)$ satisfies every inequality which $\Gamma(k,m)$ is at present known to satisfy.

2.2. *The numbers $\Delta'(k,m)$, $\delta_p(k)$, and $\lambda_p(k)$.* We can simplify our problem by further considerations which apply somewhat similarly to $\Delta(k,m)$ and $\Lambda(k,m)$. We call a solution of (1.2) *primitive*, if the greatest common divisor of $m, x_1, x_2, \ldots, x_s$ is 1. We define $\Delta'(k,m,n)$ as the least s such that (1.2) has a primitive solution for suitable $r = r(k,m,n)$, and we write

$$\Delta'(k,m) = \max \Delta'(k,m,n).$$

Let p be any prime. We write

$$k = p^\theta k^*,$$
$$(k^*,p) = 1,$$
$$\phi = \begin{cases} \theta+2 & (p=2), \\ \theta+1 & (p>2), \end{cases}$$
$$\epsilon = (p-1,k^*), \qquad p-1 = \epsilon d, \qquad k^* = \epsilon k_0.$$

Where it is not clear to which prime we refer, we write θ_p, ϕ_p, ϵ_p, d_p for θ, ϕ, ϵ, d.

We define $\gamma_p(k)$ as the least value of s such that (1.2) has a primitive solution for $m = p^\phi$, every n and $r = s$; and $\lambda_p(k)$ as the least value of s such that (1.2) has a primitive solution for $m = p^\phi$, every n and every r $(0 \leqslant r \leqslant s)$. Similarly $\delta_p(k) = \Delta'(k,p^\phi)$.

LEMMA 2. *If $(m_1,m_2) = 1$ and if there is a primitive solution of* (1.2) *for given n, s, r when $m = m_1$ and when $m = m_2$, then there is a primitive solution of* (1.2) *for the same n, s, r when $m = m_1 m_2$.*

The proof is the same as that of Lemma 1.

LEMMA 3. *If $l > \phi$ and $x = y+zp^{l-\theta-1}$, then*

$$x^k \equiv y^k + kzp^{l-\theta-1}y^{k-1} \pmod{p^l}.$$

This is Lemma 1 of P.N. 4.

LEMMA 4. *If $l > \phi$, and if there is a primitive solution of* (1.2) *for given n, s, r when $m = p^{l-1}$, then there is a primitive solution of* (1.2) *for the same n, s, r when $m = p^l$.*

The proof of this lemma is effectively the same as that of Lemma 5 of P.N. 4 and so we omit it.

LEMMA 5. $$\Lambda(k,p^t) \leqslant \lambda_p(k)$$

and $$\Delta(k,m) \leqslant \Delta'\Big(k, \prod_{p|m} p^{\phi_p}\Big).$$

The first part is obvious for $t \leqslant \phi$ and it follows for $t = \phi+1$, $\phi+2,...$ by successive applications of Lemma 4. If $m = \prod p^{t_p}$, the second part is obvious for $t_p \leqslant \phi_p$ $(p \mid m)$; for larger t_p it follows by successive applications of Lemma 4, of Lemma 2, and of the trivial converse of the latter.

LEMMA 6. (i) *Except when $k = 4$ and $p = 2$,*

$$\Lambda(k,p^k) \geqslant \lambda_p(k).$$

(ii) *Except when $k = 4$ and one of p_1, p_2,... is 2,*

$$\Delta(k, p_1^k p_2^k \dots p_i^k) \geqslant \Delta'(k, p_1^{\phi_{p_1}} p_2^{\phi_{p_2}} \dots p_i^{\phi_{p_i}}).$$

(iii) *For all k and p,*

$$\Delta(k,p^k) \geqslant \delta_p(k).$$

Apart from the case $k = 4$, $p = 2$ we have always $k > \phi$. If $s = \Lambda(k,p^k)$, then (1.2) is soluble for $m = p^k$, all values of n and every r $(0 \leqslant r \leqslant s)$. This is true in particular for $n = 1, 2, 3,..., p^\phi$. Since p^k does not divide these values of n, the solutions must be primitive and so $s \geqslant \lambda_p(k)$.

Again let $M = p_1^k \dots p_i^k$, $M' = p_1^{\phi_{p_1}} p_2^{\phi_{p_2}} \dots p_i^{\phi_{p_i}}$, and $s = \Delta(k,M)$. Then (1.2) is soluble for $m = M$, every value of n, and a suitable $r = r(k,M,n)$. If $1 \leqslant t \leqslant M'$, one of t, $t+M'$ is not divisible by p_j^k, where $1 \leqslant j \leqslant i$, since $p_j^k \nmid M'$. Hence either for $n = t$ or for $n = t+M'$ the solution of (1.2) with $m = M$ is primitive. This provides a primitive solution of (1.2) for $m = M'$ and $n = t$, since $M' \mid M$. Hence $s \geqslant \Delta'(k,M')$.

The statement (iii) is a particular case of (ii) (with $i = 1$), except when $k = 4$ and $p = 2$. But the fourth power residues (mod $2^\phi = 16$) are 0 and 1; hence

$$\delta_2(4) = 8, \qquad \Delta(4,16) = 8.$$

LEMMA 7. (i) *Except when $k = 4$ and $p = 2$,*

$$\lambda_p(k) = \max_t \Lambda(k,p^t).$$

(ii) *If $k \neq 4$ or if $k = 4$ and p_1, p_2,... are all odd, then*

$$\Delta'(k, p_1^{\phi_1} p_2^{\phi_2} \dots p_i^{\phi_i}) = \max_{t_1,t_2,\dots t_i} \Delta(k, p_1^{t_1} p_2^{t_2} \dots p_i^{t_i}).$$

(iii) *For all k and p*

$$\delta_p(k) = \max_t \Delta(k, p^t).$$

(iv)
$$\lambda_2(4) = 16, \qquad \max_t \Lambda(4, 2^t) = 15.$$

The first three results follow at once from Lemmas 5 and 6.

We see also that the congruence (1.2) has a primitive solution when $k = 4$, $m = 2^\phi = 16$, and $s = 15$ (i) for all n and $1 \leqslant r \leqslant 14$, and (ii) for all $n \not\equiv 0 \pmod{16}$ and $0 \leqslant r \leqslant 15$, but not for $r = 0$ or 15 and $n \equiv 0 \pmod{16}$. It follows that $\lambda_2(4) = 16$ and, by Lemma 4, that (1.2) has a solution with $m = 2^4$ and $s = 15$ except when $r = 0$ or 15 and $n = 16^v l$, where $v > 0$ and $l \not\equiv 0 \pmod{16}$. In the latter case there is a solution of

$$l \equiv x_1^4 + \ldots + x_r^4 - x_{r+1}^4 \ldots - x_{15}^4 \pmod{2^t}$$

for $0 \leqslant r \leqslant 15$ and so a solution of (1.2), viz.

$$n = 16^v l \equiv (2^v x_1)^4 + \ldots + (2^v x_r)^4 - (2^v x_{r+1})^4 - \ldots - (2^v x_{15})^4 \pmod{2^t}.$$

Hence $\Lambda(4, 2^t) \leqslant 15$ and so

$$\max_t \Lambda(4, 2^t) = 15,$$

since
$$\Lambda(4, 16) \geqslant \Lambda(4, 16, 15) = 15.$$

Our calculation of $\Delta(k)$ is based on the following fundamental lemma which is a consequence of the above.

LEMMA 8. *If we select a finite number of primes q, we have*

$$\Delta(k) \leqslant \max\left\{\Delta'\left(k, \prod_q q^{\phi_q}\right), \max_{p \neq q} \lambda_p(k)\right\}.$$

If $k \neq 4$,
$$\Delta(k) \geqslant \max_m \Delta'(k, m) \geqslant \max_p \delta_p(k),$$

and
$$\Delta(4) \geqslant \delta_2(4) = 8.$$

2.3. THEOREM 1. *When k is odd,*

$$\delta_2(k) = \lambda_2(k) = \gamma_2(k) = 2;$$

and, when k is even,

$$\delta_2(k) = 2^{\theta+1}, \qquad \lambda_2(k) = 2^{\theta+2}.$$

When k is odd, $\theta = 0$, $\phi = 2$, the kth-power residues to modulus 4 are 0, 1, -1, and so

$$\delta_2(k) = \lambda_2(k) = \gamma_2(k) = 2.$$

When k is even, Lemma 3 with $p = 2$, $l = \theta+3$, $y = \pm 1$ gives

$$(-1+4z)^k \equiv (1+4z)^k \equiv 1 \pmod{2^{\theta+2}}$$

and we have $$(2z)^k \equiv 0 \pmod{2^{\theta+2}},$$

since $k \geqslant \theta+2$. Hence 0 and 1 are the only kth-power residues to modulus $2^{\theta+2}$, and so

$$\delta_2(k) = 2^{\theta+1}, \qquad \lambda_2(k) = 2^{\theta+2}.$$

2.4. If -1 is a kth-power residue to modulus m, we have

$$-x_i^k \equiv y_i^k \pmod{m}$$

for some y_i, and so, if (1.2) has a solution for any one r, it has a solution for all r. Hence

$$\Delta(k,m) = \Lambda(k,m) = \Gamma(k,m).$$

This is always the case if k is odd, since $(-1)^k = -1$, and so we have

THEOREM 2. *If k is odd,*

$$\Delta(k) = \Lambda(k) = \Gamma(k).$$

Similarly, for any k, if -1 is a kth-power residue to modulus p^ϕ, we have $$\delta_p(k) = \lambda_p(k) = \gamma_p(k).$$

Henceforth we suppose $p > 2$. Then there is a primitive root G of p^ϕ and the kth-power residues to modulus p^ϕ are given by the d numbers $$G^{p^\theta\epsilon},\ G^{2p^\theta\epsilon}, \ldots,\ G^{dp^\theta\epsilon} \tag{2.41}$$

(P.N. 8, pp. 521–2). Also

$$-1 \equiv G^{\frac{1}{2}p^\theta(p-1)} = G^{\frac{1}{2}dp^\theta\epsilon} \pmod{p^\phi}.$$

Therefore -1 is a kth-power residue, if and only if

$$xp^\theta\epsilon \equiv \tfrac{1}{2}dp^\theta\epsilon \pmod{dp^\theta\epsilon}$$

is soluble, i.e. if d is even. Hence we have

LEMMA 9. *If d is even,*

$$\delta_p(k) = \lambda_p(k) = \gamma_p(k).$$

If d is odd, the kth-power residues are again given by (2.41) and their negatives by

$$G^{\frac{1}{2}p^\theta\epsilon(d+2f)} \quad (f = 1, 2, \ldots, d),$$

that is, by $$G^{\frac{1}{2}p^\theta\epsilon(1+2l)}, \tag{2.42}$$

where l runs through a complete set of residues to modulus d. The residue classes (2.41) and (2.42) together are given by the formula

$$G^{\frac{1}{2}p^{\theta}\epsilon h}, \tag{2.43}$$

where h runs through a complete set of residues to modulus $2d$.

Now let $k = 2^{\sigma}k'$ and $p-1 = 2^{\tau}\eta$, where k' and η are odd. Since d and p are odd, $\sigma \geqslant \tau \geqslant 1$. We write k_1 for $2^{\tau-1}k'$ and define θ_1, ϕ_1, ϵ_1, d_1 in terms of k_1 and p, just as we defined θ, ϕ, ϵ, d in terms of k and p. Then $\theta_1 = \theta$, $\phi_1 = \phi$,

$$\epsilon = (p-1, k^*) = 2^{\tau}(\eta, 2^{-\sigma}k^*),$$

$$\epsilon_1 = (p-1, 2^{\tau-\sigma-1}k^*) = 2^{\tau-1}(2\eta, 2^{-\sigma}k^*) = \tfrac{1}{2}\epsilon,$$

since $2^{-\sigma}k^*$ is odd, and $d_1 = 2d$. Therefore (2.41) with k_1 in place of k shows that the k_1th-power residues to modulus p^{ϕ} are

$$G^{p^{\theta}\epsilon_1 h} = G^{\frac{1}{2}p^{\theta}\epsilon h} \quad (h = 1, 2, \ldots, 2d).$$

But these are exactly the residues (2.43). Hence the problem of representing a number to modulus p^{ϕ} by a sum of positive and negative kth powers is identical with the problem of representing a number to modulus p^{ϕ} by a sum of positive k_1th powers. Combining this with Lemma 9, we have

LEMMA 10. *If $k = 2^{\sigma}k'$, $p > 2$, and $p-1 = 2^{\tau}\eta$, where k' and η are odd, then*

$$\delta_p(k) = \begin{cases} \gamma_p(k) & (\sigma < \tau), \\ \gamma_p(2^{\tau-\sigma-1}k) & (\sigma \geqslant \tau). \end{cases}$$

This reduces the calculation of $\delta_p(k)$ to that of $\gamma_p(k)$ and the latter problem has been thoroughly investigated by Hardy and Littlewood.

2.5. *Upper bounds for $\lambda_p(k)$.* As we have seen, the primitive kth-power residues to modulus p^{ϕ} are the d numbers (2.41). Hence, if $d = 1$, there is just one primitive kth-power residue, viz. 1, and if $d = 2$, there are just two, viz. 1 and -1. We deduce

THEOREM 3. *If $p > 2$ and $d = 1$,*

$$\lambda_p(k) = \gamma_p(k) = p^{\phi}, \qquad \delta_p(k) = \tfrac{1}{2}(p^{\phi}-1),$$

while, if $d = 2$,

$$\lambda_p(k) = \gamma_p(k) = \delta_p(k) = \tfrac{1}{2}(p^{\phi}-1).$$

For larger values of d we shall prove the following:

THEOREM 4. *If $p > 2$, $\theta = 0$, and $d > 1$, then*

$$\lambda_p(k) \leqslant \max(\epsilon, 3) \leqslant k.$$

THEOREM 5. *If* $p > 2$, $\theta > 0$, *and* $d > 2$, *then*

$$\lambda_p(k) \leqslant p^\theta \epsilon \leqslant k.$$

Theorems 2 and 3 of P.N. 8 give the same results for $\gamma_p(k)$. Hence, by Lemma 9, we need only prove Theorems 4 and 5 when d is odd, so that $d \geqslant 3$. Again the results for $\gamma_p(k)$ cover the case $r = s$, and the case $r = 0$ is an immediate corollary. Hence, to complete the proofs of Theorems 4 and 5, it is enough to prove that, when $m = p^\phi$, d is odd, and $s = p^\theta \epsilon$, then (1.2) has a primitive solution for every r such that $1 \leqslant r \leqslant s-1$ and for all n. In fact, if $n \not\equiv 0 \pmod{p^\phi}$, any solution must be primitive, since p^ϕ is a divisor of p^k; while, if $n \equiv 0$, the primitive solution $0 = 1^k - 1^k$ is admissible, since

$$1 \leqslant r \leqslant s-1.$$

Hence we see that Theorems 4 and 5 follow from the following two lemmas.

LEMMA 11. *If* $p > 2$, *then*

$$\Lambda(k,p) \leqslant \epsilon \leqslant k.$$

LEMMA 12. *If* $p > 2$, $d \geqslant 3$, *and* $\theta > 0$, *then*

$$\Lambda(k,p^\phi) \leqslant p^\theta \epsilon \leqslant k.$$

The proof of Lemma 16 of P.N. 8 can be generalized without difficulty to prove our Lemma 11, but it is simpler to use a theorem due to Davenport and I. Chowla,† viz.:

Let $\{a\} = \{a_1, a_2, \ldots, a_n\}$ *and* $\{b\} = \{b_1, b_2, \ldots, b_n\}$ *be two different systems of residue classes to modulus* h. *No two* a *are congruent to modulus* h *and no two* b *are congruent to modulus* h. *Also* $(b_i, h) = 1$. *Let* l *be the number of different residue classes* c_l *to modulus* h *of the form* a_i *or* $a_i + b_j$. *Then*

$$l \geqslant \min\{h, m+n\}.$$

We take $h = p$. To prove that (1.2) is soluble with $m = p$, $s = \epsilon$, and a fixed $r \leqslant s$, we first apply the theorem r times in succession, starting with $\{a\} = \{0\}$ and $\{b\} = \{x\}$, where x runs through a complete set of primitive kth-power residues to modulus p, and continuing with the same set $\{b\}$, but taking as $\{a\}$ the sum-set $\{c\}$ obtained at the previous application. Next we apply the theorem

† Landau, *Über einige neuere Ergebnisse der additiven Zahlentheorie* (Cambridge, 1937), Satz 114.

again $\epsilon - r$ times, taking $\{b\} = \{-x\}$, where x runs through a complete set of primitive kth-power residues to modulus p. We find finally that

$$\epsilon d+1 = p-1+1 = p$$

residues, i.e. all residues to modulus p, are representable in the form

$$\sum_{i=1}^{r} x_i^k - \sum_{i=r+1}^{\epsilon} x_i^k.$$

Since this is true for all $r \leqslant \epsilon$, Lemma 11 is proved.

2.6. *Proof of Lemma* 12. We base the proof of Lemma 12 on a series of lemmas.

We write $N(s,r)$ for any n which can be expressed in the form (1.1) and we use

$$n \equiv N(s,r) \pmod{m}$$

to denote that (1.2) is soluble for the particular m, n, s, r concerned. It is clear that

$$N(s_1,r_1)+N(s_2,r_2) = N(s_1+s_2, r_1+r_2),$$

and
$$lN(s,r) = N(ls, lr).$$

In this section all congruences are to modulus p^2 except when it is stated otherwise.

LEMMA 13. *If $d \geqslant 3$, then there is a kth-power residue c to modulus p^2 such that*

$$(p,c) = 1, \qquad p+1 \leqslant c \leqslant (p-1)p-1, \qquad c \neq \tfrac{1}{2}(p^2+1).$$

Since $d \geqslant 3$, there is at least one kth-power residue not divisible by p and not congruent to 1 or -1, say x^k. Then there is a number y such that $xy \equiv 1$, so that $x^k y^k \equiv 1$ and $y^k \not\equiv 1$. Hence, if $x^k \equiv a$ and $y^k \equiv b$, where $|a| < \frac{1}{2}p^2$ and $|b| < \frac{1}{2}p^2$, we have $|a| > 1$ and $|b| > 1$. Also, if

$$|a| \leqslant p-1, \qquad |b| \leqslant p-1,$$

we have
$$1 < |ab| < (p-1)^2,$$

so that
$$x^k y^k \equiv ab \not\equiv 1,$$

a contradiction. Hence either $|a| > p$ or $|b| > p$. Let us suppose that $|a| > p$; then, if $x^k \not\equiv \frac{1}{2}(p^2+1)$, we can take $c \equiv a \equiv x^k$, where $p+1 \leqslant c \leqslant (p-1)p-1$.

If $x^k \equiv \frac{1}{2}(p^2+1)$, we have

$$(x^2)^k \equiv \tfrac{1}{4}(p^4+2p^2+1) = p^2\tfrac{1}{4}(p^2-1)+\tfrac{1}{4}(3p^2+1)$$
$$\equiv \tfrac{1}{4}(3p^2+1).$$

Since $d \geqslant 3$, $p \geqslant 5$ and so

$$p+1 < \tfrac{1}{4}(3p^2+1) \leqslant (p-1)p-1.$$

Hence in this case we can take $c = \frac{1}{4}(3p^2+1) \neq \frac{1}{2}(p^2+1)$.

LEMMA 14. *If $d \geqslant 3$, then for any r $(0 \leqslant r \leqslant p-1)$ we can find $u = u(k,p,r) \not\equiv 0 \pmod p$ such that*

$$up \equiv N(p-1,r) \pmod{p^2}. \tag{2.61}$$

We write $r' = p-1-r$. The case $r = p-1$ is Lemma 17 of P.N. 8. Hence we may suppose $r \leqslant p-2$ and $r' \geqslant 1$. We may further suppose $r \geqslant r'$ since otherwise we have only to change u into $-u$. Since $p \geqslant 5$, we have

$$r \geqslant \tfrac{1}{2}(p-1) \geqslant 2.$$

Using the c of Lemma 13, we write

$$\mu = \left[\frac{c}{p}\right]+1, \qquad \mu p = c+h.$$

Then

$$(\mu-1)p+1 \leqslant c \leqslant \mu p-1$$

$$2 \leqslant \mu \leqslant p-1, \qquad 1 \leqslant h \leqslant p-1.$$

If $h < r$, we have

$$\mu p = c+h \equiv N(h+1,h+1) = N(p-1,r)$$

and $\mu \not\equiv 0 \pmod p$; again, if $h > r' \geqslant 1$, i.e. $p-h \leqslant r$,

$$(1-\mu)p = -c+(p-h) \equiv N(p-h+1,p-h) = N(p-1,r)$$

and $1-\mu \not\equiv 0 \pmod p$. The only remaining case is

$$h = r' = r = \tfrac{1}{2}(p-1).$$

In this case

$$(2\mu-1)p = 2c-1 \equiv N(3,2) = N(p-1,r),$$

since $r \geqslant 2$ and $r' \geqslant 1$. This proves the lemma, unless

$$2\mu-1 \equiv 0 \pmod p, \quad \text{i.e. } \mu = \tfrac{1}{2}(p+1).$$

But this implies

$$c = \mu p-h = \tfrac{1}{2}(p^2+1)$$

in contradiction to Lemma 13.

LEMMA 15. *For any integers n and ρ $(0 \leqslant \rho \leqslant p-1)$*

$$np \equiv N(\epsilon(p-1), \epsilon\rho) \pmod{p^2}.$$

By Lemma 14 we can find $u \not\equiv 0 \pmod p$ such that

$$up \equiv N(p-1,\rho) = \sum_{i=1}^{\rho} y_i^k - \sum_{i=\rho+1}^{p-1} y_i^k.$$

Since $u \not\equiv 0$, we can find v such that $uv \equiv n \pmod p$. By Lemma 11,

$$v \equiv N(\epsilon, \epsilon) \equiv \sum_{h=1}^{\epsilon} x_h^k \pmod p.$$

Hence

$$\begin{aligned} np \equiv uvp &\equiv \sum_{h=1}^{\epsilon} x_h^k up \\ &\equiv \sum_{h=1}^{\epsilon} x_h^k \Big(\sum_{i=1}^{\rho} y_i^k - \sum_{i=\rho+1}^{p-1} y_i^k\Big) \\ &\equiv \sum_{i=1}^{\rho} \sum_{h=1}^{\epsilon} (x_h y_i)^k - \sum_{i=\rho+1}^{p-1} \sum_{h=1}^{\epsilon} (x_h y_i)^k \\ &\equiv N(\epsilon(p-1), \epsilon\rho). \end{aligned}$$

We now deduce Lemma 12. We have to consider (1.2) with $s = p^\theta \epsilon$, $m = p^\phi$, and $0 \leqslant r \leqslant p^\theta \epsilon$. We can write r in the form

$$r = \rho_0 + \epsilon(\rho_1 + \rho_2 p + \rho_3 p^2 ... + \rho_\theta p^{\theta-1}),$$

where $\quad 0 \leqslant \rho_0 \leqslant \epsilon, \qquad 0 \leqslant \rho_i \leqslant p-1 \quad (i = 1, 2, ..., \theta).$

For any n we have, by Lemma 11 and repeated applications of Lemma 15,

$$\begin{aligned} n &= N(\epsilon, \rho_0) + n_1 p \\ &= N(\epsilon, \rho_0) + N(\epsilon(p-1), \epsilon\rho_1) + n_2 p^2 \\ &= N(\epsilon, \rho_0) + N(\epsilon(p-1), \epsilon\rho_1) + pN(\epsilon(p-1), \epsilon\rho_2) + n_3 p^3 \\ &= ... \\ &= N(\epsilon, \rho_0) + N(\epsilon(p-1), \epsilon\rho_1) + pN(\epsilon(p-1), \epsilon\rho_2) + \\ &\qquad + p^2 N(\epsilon(p-1), \epsilon\rho_3) + ... + p^{\theta-1} N(\epsilon(p-1), \epsilon\rho_\theta) + n_{\theta+1} p^{\theta+1} \\ &\equiv N(\epsilon + \epsilon(p-1) + p\epsilon(p-1) + ... + p^{\theta-1}\epsilon(p-1), \\ &\qquad \rho_0 + \epsilon\rho_1 + \epsilon\rho_2 p + ... + \epsilon\rho_\theta p^{\theta-1}) \pmod{p^\phi} \\ &\equiv N(\epsilon p^\theta, r) \pmod{p^\phi}. \end{aligned}$$

This is Lemma 12.

2.7. We may sum up certain of our results for $\delta_p(k)$ and $\lambda_p(k)$ as follows:

LEMMA 16. (i) *If* $\vartheta > 1$,

$$\lambda_2(2^\vartheta) = \lambda_2(2^\vartheta.3) = 2^{\vartheta+2}, \qquad \delta_2(2^\vartheta) = \delta_2(2^\vartheta.3) = 2^{\vartheta+1}.$$

(ii) *If* π *is an odd prime and* $\vartheta \geqslant 0$, *then*

$$\lambda_\pi(\pi^\vartheta(\pi-1)) = \pi^{\vartheta+1},$$

$$\delta_\pi(\pi^\vartheta(\pi-1)) = \lambda_\pi(\tfrac{1}{2}\pi^\vartheta(\pi-1)) = \delta_\pi(\tfrac{1}{2}\pi^\vartheta(\pi-1)) = \tfrac{1}{2}(\pi^{\vartheta+1}-1).$$

(iii) *In all other cases*, $\delta_p(k) \leqslant \lambda_p(k) \leqslant k$.

The results (i) and (ii) are immediate consequences of Theorems 1 and 3; (iii) follows from Theorems 1, 4, 5, except when $p > 2$ and

$d = 1$ or 2. If $d = 1$ and $k \neq p^\theta(p-1)$, we have $p^\theta(p-1) \mid k$ and $k \geqslant 2p^\theta(p-1)$. Hence

$$\lambda_p(k) = p^{\theta+1} < 2(p^{\theta+1}-p^\theta) \leqslant k.$$

If $d = 2$ and $k \neq \frac{1}{2}p^\theta(p-1)$, we have $\frac{1}{2}p^\theta(p-1) \mid k$ and $k \geqslant p^\theta(p-1)$. Hence

$$\lambda_p(k) = \tfrac{1}{2}(p^{\theta+1}-1) < p^\theta(p-1) < k.$$

We shall also require the following lemmas.

LEMMA 17. *If* $(m_1, m_2) = 1$, $m_1 > m_2 > 2$ *and* m_1 *is odd, then*

$$\Delta'(k, m_1 m_2) \leqslant \max\{\tfrac{1}{2}(m_1+1), m_2-1\}.$$

Since 1 is a kth-power residue to modulus m_1 and to modulus m_2, we see that (1.2) has a primitive solution for every n (i) with $m = m_2$, $s = m_2-1$, $r = m_2-2$, (ii) with $m = m_2$, $s = m_2-1$, $r = 1$, and (iii) with $m = m_1$, $s = \frac{1}{2}(m_1+1)$, and $r = \frac{1}{2}(m_1-1)$ or $r = 1$ according to the value of n. In particular the residue 0 is always representable as 1^k-1^k. Hence, by Lemma 2, (1.2) has a primitive solution with $m = m_1 m_2$, $s = \max\{\frac{1}{2}(m_1+1), m_2-1\}$, and $r = s-1$ or $r = 1$ according to the value of n. This is our lemma.

LEMMA 18. *If* $\theta' > 0$ *and* $\theta'' > 0$, *then*

$$\begin{aligned}
p_1^{\theta'}(p_1-1) &\neq p_2^{\theta''}(p_2-1), && \textit{unless } p_1 = p_2,\\
\tfrac{1}{2}p_1^{\theta'}(p_1-1) &\neq p_2^{\theta''}(p_2-1), && \textit{unless } p_1 = p_2 = 2,\\
2^{\theta'}.3 &\neq p^{\theta''}(p-1), && \textit{unless } p = 3,\ \theta' = \theta'' = 1,\\
2^{\theta'}.3 &\neq \tfrac{1}{2}p^{\theta''}(p-1).
\end{aligned}$$

The last two statements are obvious, while, if either of the first two were false, we should have $p_1^{\theta'} \mid p_2-1$ and $p_2^{\theta''} \mid p_1-1$, so that $p_1 < p_2$ and $p_2 < p_1$, a contradiction.

In our numerical calculations we shall use the following lemma. We omit the proof, which is a simple adaptation of those of Lemmas 11 and 12 of P.N. 8.

LEMMA 19. *If* $\theta = 0$, $s > 2$ *and*

$$p > (\epsilon-1)^{2(s-1)/(s-2)},$$

then $\lambda_p(k) \leqslant s$.

2.8. *General results for* $\Delta(k)$. We can now find the actual values of $\Delta(k)$ when k belongs to one of the following classes:

(I) $k = 2^\vartheta$ $(\vartheta > 1)$.

(II) $k = \frac{1}{2}\pi^\vartheta(\pi-1)$, π an odd prime, $\vartheta > 0$, and k not in class I.

(III) $k = \frac{1}{2}(\pi-1)$, π an odd prime. and k not in class I or class II.

THEOREM 6.† (i) *If k belongs to class* I, $\Delta(k) = 2^{\vartheta+1}$.

(ii) *If k belongs to class* II, $\Delta(k) = \frac{1}{2}(\pi^{\vartheta+1}-1) \geqslant k+1$.

(iii) *If k does not belong to classes* I *or* II, $\Delta(k) \leqslant k$.

(iv) *If k belongs to class* III, $\Delta(k) = \frac{1}{2}(\pi-1) = k$.

Let $k = 2^{\vartheta}$ ($\vartheta > 1$); then $\delta_2(k) = 2k$ by Theorem 1. If $p > 2$, then $\theta = 0$. If $d = 1$, then

$$(p-1) \mid k, \qquad p-1 \leqslant k, \qquad \lambda_p(k) = p \leqslant k+1 < 2k$$

by Theorem 3. If $d \geqslant 2$, $\lambda_p(k) \leqslant k$ by Theorem 4. Hence, by Lemma 8,

$$\Delta(k) \leqslant \max\left\{\delta_2(k), \max_{p>2} \lambda_p(k)\right\} = 2k$$

and

$$\Delta(k) \geqslant \delta_2(k) = 2k.$$

Hence we have (i).

We say that k is exceptional with respect to 2 if $k = 2^{\vartheta}.3$ ($\vartheta > 1$) and that k is exceptional with respect to the odd prime π if k is not a power of 2 and if either

$$k = \pi^{\vartheta}(\pi-1) \quad (\vartheta \geqslant 0) \quad \text{or} \quad k = \tfrac{1}{2}\pi^{\vartheta}(\pi-1) \quad (\vartheta \geqslant 1).$$

Lemma 16 shows that $\lambda_p(k) \leqslant k$ unless k is a power of 2 or k is exceptional with respect to p. We have to study $\Delta'(k, \prod q^{\phi_q})$ for exceptional k, the product being taken over all primes q with respect to which k is exceptional.

If k is exceptional with respect to one prime only, either

$$k = 2^{\vartheta}.3 \quad (\vartheta > 1), \qquad \Delta'(k, \textstyle\prod q^{\phi_q}) = \delta_2(k) = 2^{\vartheta+1} < k,$$

or
$$k = \pi^{\vartheta}(\pi-1) \quad (\vartheta \geqslant 0), \qquad \delta_\pi(k) = \tfrac{1}{2}(\pi^{\vartheta+1}-1) < k,$$

or
$$k = \tfrac{1}{2}\pi^{\vartheta}(\pi-1) \quad (\vartheta > 0), \qquad \delta_\pi(k) = \tfrac{1}{2}(\pi^{\vartheta+1}-1) \geqslant k+1.$$

If k is exceptional with respect to each of two primes, either

$$k = 2^{\vartheta}.3 = \pi-1 \quad (\vartheta > 1),$$
$$\Delta'(k, 2^{\vartheta+1}\pi) \leqslant \max\{\tfrac{1}{2}(\pi+1, 2^{\vartheta+1}-1\} \leqslant k,$$

or
$$k = \pi^{\vartheta}(\pi-1) = \varpi-1 \quad (\vartheta > 0),$$
$$\Delta'(k, \pi^{\vartheta+1}\varpi) \leqslant \max\{\tfrac{1}{2}(\pi^{\vartheta+1}+1), \varpi-1\} \leqslant k,$$

or $k = \frac{1}{2}\pi^{\vartheta}(\pi-1) = \varpi-1$ ($\vartheta > 0$), $\delta_\pi(k) = \frac{1}{2}(\pi^{\vartheta+1}-1) \geqslant k+1$,

and
$$\lambda_\varpi(k) = \varpi = k+1,$$

† This includes two theorems stated without proof by Wright, *Quart. J. of Math.* (Oxford), 6 (1935), 266.

by Lemmas 16 and 18. Finally, by Lemma 18, the only k exceptional with respect to more than two primes is 6; and

$$\Delta'(6, 2^3.3^2.7) \leqslant 6$$

by numerical calculation. Hence, by Lemma 8,

$$\Delta(k) \leqslant k,$$

except when $k = \frac{1}{2}\pi^{\vartheta}(\pi-1)$ $(\vartheta > 0)$, when

$$\Delta(k) = \delta_{\pi}(k) = \tfrac{1}{2}(\pi^{\vartheta+1}-1).$$

To prove (iv) we have (iii) and

$$\Delta(\tfrac{1}{2}(\pi-1)) \geqslant \delta_{\pi}(\tfrac{1}{2}(\pi-1)) = \tfrac{1}{2}(\pi-1).$$

2.9. *Calculation of $\Delta(k)$ for particular k.* With the aid of the general results proved above, we have been able to calculate $\Delta(k)$ for values of $k \leqslant 36$ without excessive labour. Most of the remarks in P.N. 8 concerning the calculation of $\Gamma(k)$ apply, *mutatis mutandis*, to that of $\Delta(k)$.

If k is odd, Theorem 2 enables us to obtain the value of $\Delta(k)$ from that of $\Gamma(k)$ calculated by Hardy and Littlewood. Theorem 6 gives the value of $\Delta(k)$ when $k = 4$, 8, 16, 32 (class I), 10 (class II), and 6, 14, 18, 20, 26, 30, 36 (class III). The remaining values of k below 36 are

$$12,\ 22,\ 24,\ 28,\ 34.$$

As an example of the calculation of $\Delta(k)$ for these k we give in detail that of $\Delta(12)$. We consider first for which primes q it is likely that $\lambda_q(12)$ will be largest. By Theorem 4, we may expect to find these among the p for which $\theta > 0$ or for which $\epsilon = k = 12$, i.e. the primes $lk+1$. Thus the least primes we have to consider are $q = 2$, 3, 13. For the three primes 2, 3, 13, we have $d = 1$, so that 0 and 1 are the only twelfth-power residues to modulus 16, 9, and 13. It is easily seen that

$$\Delta(12) \geqslant \Delta'(12, 9.16) \geqslant \Delta'(12, 9.16, 72) = 9. \qquad (2.91)$$

Again every residue to modulus $9.13.16$ has a primitive representation with $s = 9$ and r suitably chosen among the values 0, 1, 3, 6, 8, 9. We note that either $r \geqslant 6$ or $s-r \geqslant 6$, so that there are always either six positive or six negative twelfth powers available.

If $p > 3$ and $p \not\equiv 1 \pmod{12}$, we have

$$\lambda_p(12) \leqslant \max(\epsilon, 3) \leqslant 6 < 9$$

by Theorem 4. Also $\lambda_p(12) \leqslant 9$ for

$$p > 11^{16/7}, \quad \text{i.e. } p > 240$$

by Lemma 19. Hence, if we take the primes q of Lemma 8 to be

$$2,\ 3,\ 13,\ 37,\ 61,\ 73,\ 97,\ 109,\ 157,\ 181,\ 193,\ 229, \tag{2.92}$$

i.e. the primes 2, 3 and those primes of the form $12l+1$ which are less than 240, we have

$$\max_{p\neq q} \lambda_p(12) \leqslant 9. \tag{2.93}$$

By calculation we find that $\gamma_q(12) \leqslant 6$ for all the primes of (2.92) except 2, 3, 13; that is, (1.2) has a primitive solution for (i) $m = \prod_{q>13} q^{\phi_q}$, $s = 6$, $r = 6$, every n, and (ii) the same m, $s = 6$, $r = 0$, every n. As we saw above, (1.2) has a primitive solution for $m = 2^4.3^2.13$, $s = 9$, every n, either $r \geqslant 6$ or $r \leqslant 3$, and so, by Lemma 2, for $m = \prod_q q^{\phi_q}$, $s = 9$, every n, and suitable r. Hence

$$\Delta'\Big(12, \prod_q q^{\phi_q}\Big) \leqslant 9. \tag{2.94}$$

By (2.93), (2.94), and Lemma 8, $\Delta(12) \leqslant 9$. Hence, by (2.91),

$$\Delta(12) = 9.$$

Here we have an example of $\Delta(k)$ greater than $\max_p \delta_p(k)$, for $\max_p \delta_p(12) = \delta_2(12) = 8$.

TABLE 1

k	3	4	5	6	7	8	9	10	11	12	13	
Δ	4	8	5	6	4	16	13	12	11	9	6	
k	14	15	16	17	18	19	20	21	22	23	24	
Δ	14	15	32	6	18	4	20	24	11	23	16	
k	25	26	27	28	29	30	31	32	33	34	35	36
Δ	10	26	40	15	29	30	5	64	33	10	35	36

3. Upper and lower bounds for $v(k)$.

We write

$$[a_1,\ldots, a_j]_{k-2} = [a'_1,\ldots, a'_j]_{k-2} \tag{3.1}$$

to denote that

$$\sum_{i=1}^{j} a_i^l = \sum_{i=1}^{j} a_i'^l \quad (l = 1, 2,\ldots, k-2),$$

$$\sum_{i=1}^{j} a_i^{k-1} \neq \sum_{i=1}^{j} a_i'^{k-1}.$$

It is obvious that (3.1) is equivalent to the identity (1.3) with

$$C = C_k = k\Big\{\sum_{i=1}^{j} a_i^{k-1} - \sum_{i=1}^{j} a_i'^{k-1}\Big\}. \tag{3.2}$$

Tarry† has remarked that (3.1) implies that

$$[a_1,..., a_j, a'_1+h,..., a'_j+h]_{k-1} = [a'_1,..., a'_j, a_1+h,..., a_j+h]_{k-1} \quad (3.3)$$

for every h; this provides an identity of the type (1.3) with $k+1$ replacing k. There is a simple relation between the new $C = C_{k+1}$ and the C_k of the first identity. Thus

$$\begin{aligned} C_{k+1} &= (k+1)\sum_{i=1}^{j}\{a_i^k+(a'_i+h)^k-a_i'^k-(a_i+h)^k\} \\ &= (k+1)kh\sum_{i=1}^{j}\{a_i'^{k-1}-a_i^{k-1}\} \\ &= -(k+1)hC_k. \end{aligned} \quad (3.4)$$

In the special case $j = k-1$, we can express C_k in terms of the a and the a' in a form which shortens its calculation. If

$$A_1 = -\sum a_i, \qquad A_2 = \sum a_{i_1} a_{i_2}, \quad ...$$

are the symmetric functions of the a, and A'_1, A'_2,... the corresponding functions of the a', and if

$$S_l = \sum_{i=1}^{j} a_i^l, \qquad S'_l = \sum_{i=1}^{j} a_i'^l,$$

then Girard and Newton‡ proved that

$$S_l+A_1S_{l-1}+2A_2S_{l-2}+...+lA_l = 0 \qquad (l = 1, 2,..., j)$$

and similarly for the A' and the S'. Taking $j = k-1$, we see that $A_1 = A'_1$, $A_2 = A'_2$,..., $A_{k-2} = A'_{k-2}$ follow from (3.1). Hence

$$\begin{aligned} C_k &= k\{S_{k-1}-S'_{k-1}\} = k(k-1)(A'_{k-1}-A_{k-1}) \\ &= (-1)^k k(k-1)\Big\{\prod_{i=1}^{k-1} a_i - \prod_{i=1}^{k-1} a'_i\Big\}. \end{aligned} \quad (3.5)$$

For $k = 7$ we start from $[0, 2]_1 = [1, 1]_1$, use Tarry's device four times taking $h = 2$, 3, 4, 5 in succession, and obtain

$$[0, 3, 5, 11, 13, 16]_5 = [1, 1, 8, 8, 15, 15]_5. \quad (3.6)$$

For $k = 8$ we use the solution of (3.1), viz.

$$[0, 18, 27, 58, 64, 89, 101]_6 = [1, 13, 38, 44, 75, 84, 102]_6, \quad (3.7)$$

discovered by Escott (and later independently by Phillips). For $k = 9$ we use the solution of (3.1), viz.

$$[0, 4, 9, 23, 27, 41, 46, 50]_7 = [1, 2, 11, 20, 30, 39, 48, 49]_7, \quad (3.8)$$

† *L'intermédiaire des mathématiciens*, 19 (1912), 200, 219–21; 20 (1913), 68–70.

‡ I. Newton, *Opera omnia*, ed. Horsley (London, 1779), 1, 182.

discovered by Tarry. In each of them $j = k-1$, and we can calculate C_k very simply from (3.5).

From (3.8) we find solutions of (3.1) for $k = 10$ and for $12 \leqslant k \leqslant 20$ by using Tarry's device with suitable h. For $k = 11$ we start from $[0,3]_1 = [1,2]_1$ and apply the device repeatedly. The values of h used are given in Table 2, as are the values of C which are found by the repeated use of (3.4).

For $k = 6$ we can improve upon the upper bound obtained from the identities of the type (1.3) by using a slightly different type of identity, viz.

$$12abcdfgx = (a^5c+bdx)^6+(a^5d-bcx)^6+(b^5c-adx)^6+(b^5d+acx)^6-$$
$$-(a^5c-bdx)^6-(a^5d+bcx)^6-(b^5c+adx)^6-(b^5d-acx)^6, \quad (3.9)$$

where $f = c^4-d^4$ and $g = a^{24}-b^{24}$. This identity is due to Subba Rao. We have not been able to find similar suitable identities for higher values of k.

If we take a, b, c, d positive integers, it is easy to see that (3.9) leads to

$$v(k) \leqslant 2j+\Delta(6, 12abcdfg),$$

where $2j = 8$ is the number of powers on the right-hand side of (3.9).

TABLE 2†

k	*Identity*	j	$\lvert C_k \rvert$
6	(3.9)	4	$12abcdfg$
7	(3.6)	6	$6.7.8^2.15^2$
8	(3.7)	7	$7.8.13.38.44.75.84.102$
9	(3.8)	8	$2^{11}.3^5.5^2.7^2.11.13$
10	(3.8); $h = 9$	10	$9.10\lvert C_9\rvert$
11	$[0, 3]_1 = [1, 2]_1$; $h = 3, 4, 5, 7, 9, 11, 13, 17$	12	$2^{10}.3^7.5^3.7^2.11^2.13.17$
12	Identity of $k = 10$; $h = 17, 19$	14	$11.12.17.19\lvert C_{10}\rvert$
13	,, ,, $k = 12$; $h = 6$	18	$6.13\lvert C_{12}\rvert$
14	,, ,, $k = 13$; $h = 4$	24	$4.14\lvert C_{13}\rvert$
15	,, ,, $k = 14$; $h = 15$	32	$15^2\lvert C_{14}\rvert$
16	,, ,, $k = 15$; $h = 10$	33	$16.10\lvert C_{15}\rvert$
17	,, ,, $k = 14$; $h = 1, 7, 10$	34	$7.10.15.16.17\lvert C_{14}\rvert$
18	,, ,, $k = 17$; $h = 15$	46	$18.15\lvert C_{17}\rvert$
19	,, ,, $k = 18$; $h = 23$	56	$19.23\lvert C_{18}\rvert$
20	,, ,, $k = 19$; $h = 16$	60	$20.16\lvert C_{19}\rvert$

† The identity for $k = 6$ was given by K. Subba Rao, *J. of London Math. Soc.* 13 (1938), 14–16; that for $k = 8$ by Escott, *Quart. J. of Math.* 41 (1910), 152; that for $k = 9$ by Tarry, loc. cit.; those for $k = 11$, 17–20 by S. Sastry and T. Rai, *Indian Phys. Math. J.* 7 (1936), 17–19; those for $k = 12$–16 by A. Moessner and W. Schulz, *Math. Zeits.* 41 (1936), 340–4.

It remains to calculate $\Delta(k, C_k)$.

[$k = 6$.] It is easy to see that $C_6 = 12abcdkl$ is always divisible by $7.8.9$. Hence

$$6 = \Delta(6, 7.8.9) \leqslant \Delta(6, C_6) \leqslant \Delta(6) = 6.$$

[$k = 7$.] We have $C_7 = 6.7.8^2.15^2$. But since

$$\phi = 2 \quad (p = 2), \qquad \phi = 1 \quad (p = 3, 5)$$

for $k = 7$, it is sufficient to consider primitive solutions of congruences to modulus $2^2.3.5.7$. Since

$$d = 2 \quad (p = 3), \qquad d = 4 \quad (p = 5), \qquad d = 6 \quad (p = 7),$$

$$\delta_2(7) = 2,$$

every number is a 7th-power residue to modulus 3, 5, and 7, and two powers are sufficient to represent any residue to modulus $2^2.3.5.7$. Hence

$$\Delta(7, C_7) = 2.$$

[$k = 8$.] Since $2^\phi = 32$ is a divisor of C_8,

$$16 = \delta_2(8) \leqslant \Delta(8, C_8) \leqslant \Delta(8) = 16.$$

[$k = 9$.] Since $27 = 3^\phi$ is a divisor of C_9,

$$13 = \delta_3(9) \leqslant \Delta(9, C_9) \leqslant \Delta(9) = 13.$$

[$k = 10$.] $5^\phi = 25$ divides C_{10} and $\delta_5(10) = 12$ since $d = 2$. Hence

$$12 = \delta_5(10) \leqslant \Delta(10, C_{10}) \leqslant \Delta(10) = 12.$$

[$k = 11$.] If $p \mid C_{11}$ and $p \neq 11$, we have $\theta = 0$, $\epsilon = 1$, and therefore $\lambda_p(11) \leqslant 3$ by Theorem 4. The 11th-power residues to modulus 11^2 are ± 1, ± 3, ± 9, ± 27, ± 81, and we find that $\delta_{11}(11) = \lambda_{11}(11) = 4$. Also $11^2 \mid C_{11}$ and so

$$4 = \delta_{11}(11) \leqslant \Delta(11, C_{11}) \leqslant \max_{p \mid C_k} \lambda_p(11) = 4$$

by (2.12) and Lemma 7.

Similar calculations determine the values of the remaining $\Delta(k, C_k)$. In the third column of Table 3 we give the least M for which

$$\Delta(k, M) = \Delta(k, C_k).$$

TABLE 3

k	$\Delta(k, C_k)$	M	*Lower bound for* $v(k) = \Delta(k)$	*Upper bound for* $v(k) = 2j + \Delta(k, C_k)$
6	6	$7.8.9$	6	14
7	2	$2^2.3.5.7$	7	14
8	16	32	16	30
9	13	27	13	29
10	12	25	12	32
11	4	11^2	11	28
12	10	$16.9.13$	9	38
13	3	13^2	6	39
14	5	8.7^2	14	53
15	5	5^2	15	69
16	32	64	32	98
17	4	17^2	6	72
18	15	$3^3.19$	18	107
19	3	19^2	4	115
20	13	16.25	20	133

SOME GENERALIZATIONS OF A FORMULA OF RAMANUJAN

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1. In a paper published in 1937 in *The Quarterly Journal of Mathematics*,* Hardy discussed the validity of two formulae of Ramanujan. These gave the cosine or sine transforms of a function defined by an infinite series, and a suggestion was made that they might be extended to the more general Watson transforms, with a formal indication of the method to be used.

In the following, Theorem 1 is a more symmetrical and slightly more general form of the theorem for cosine transforms proved by Hardy. Theorem 2 is an extension to general Watson transforms, and Theorem 3 is a discussion of a special case closely related to a Hilbert transform.

Theorem 1. *Let $\chi(u)$ be an integral function satisfying the condition*

$$\frac{\chi(\sigma+it)}{2^{\frac{1}{2}|\sigma|}\Gamma(\frac{1}{2}|\sigma|+\frac{1}{2}it)} = O(e^{A|\sigma|+B|t|}),$$

where $B < \pi$, for all σ and t. Then

$$\sum_{0}^{\infty} \frac{(-1)^n\chi(n)}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+\frac{1}{2})}x^n$$

is convergent for $0 \leqslant x < e^{-A}$, and represents an analytic function $L(x)$ regular for all positive x. Also

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_{0}^{\infty} L(x)\cos yx\,dx = M(y),$$

where
$$M(y) = \sum_{0}^{\infty} \frac{(-1)^n\chi(-n-1)}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+\frac{1}{2})}y^n$$

for $0 \leqslant y < e^{-A}$, and is the analytic continuation of this for all positive y. Also

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_{0}^{\infty} M(x)\cos yx\,dx = L(y).$$

* G. H. Hardy, 'Ramanujan and the theory of Fourier transforms': *Quart. J. of Math.* (Oxford), 8 (1937), 245–54.

In view of the similarity of this theorem to the theorem proved by Hardy, I will only sketch the proof. Suppose that

$$-1 < c < 0, \qquad 0 < x < e^{-A}.$$

Then an application of Cauchy's theorem shows that

$$L(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi\chi(u)x^u\,du}{2^{\frac{1}{2}u}\Gamma(\frac{1}{2}u+\frac{1}{2})\sin\pi u},$$

where x^u has its principal value. This is true at first for $0 < x < e^{-A}$, but since the integral is uniformly convergent in any interval $0 < \delta \leqslant x \leqslant \Delta < \infty$, it gives the continuation of $L(x)$ for all positive x.

It follows that, if y is real and positive,

$$\begin{aligned}
&\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty L(x)\cos yx\,dx \\
&\qquad = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \cos yx\left(-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi\chi(u)x^u\,du}{2^{\frac{1}{2}u}\Gamma(\frac{1}{2}u+\frac{1}{2})\sin\pi u}\right)dx \\
&\qquad = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi\chi(u)\,du}{2^{\frac{1}{2}u}\Gamma(\frac{1}{2}u+\frac{1}{2})\sin\pi u}\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty x^u\cos yx\,dx \\
&\qquad = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi\chi(u)y^{-u-1}\,du}{2^{-\frac{1}{2}(u+1)}\Gamma(-\frac{1}{2}u)\sin\pi u}.
\end{aligned}$$

The inversion of the order of integration may be justified by a method exactly similar to that used in Hardy's paper originally referred to, and so need not be dealt with here.

Another application of Cauchy's theorem gives

$$\begin{aligned}
\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty L(x)\cos yx\,dx &= -\sum_1^\infty \frac{(-1)^n\chi(-n)}{2^{\frac{1}{2}n-\frac{1}{2}}\Gamma(\frac{1}{2}n)}y^{n-1} \\
&= \sum_0^\infty \frac{(-1)^n\chi(-n-1)}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+\frac{1}{2})}y^n = M(y).
\end{aligned}$$

This again holds in the first place for $0 < y < e^{-A}$, but since the integral representation is uniformly convergent for

$$0 < \delta \leqslant y \leqslant \Delta < \infty,$$

the final result holds for all positive y.

In the above, $\chi(u)$ satisfies conditions symmetrical about the origin, so we may replace $\chi(u)$ by $\chi(-u-1)$. This merely reverses the roles of $L(x)$ and $M(x)$, and so we obtain the reciprocal formula

$$\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty M(x)\cos yx\,dx = L(y).$$

2. Examples. I select two examples which are not covered by Hardy's theorem. Let

$$\chi(u) = \frac{2^{\frac{1}{2}u}\Gamma(\frac{1}{2}u+\frac{1}{2})}{\Gamma(n+1)}.$$

The conditions of Theorem 1 are satisfied with $A = 0$, $B = \frac{1}{2}\pi$, and we obtain as reciprocal functions

$$L(x) = e^{-x}, \qquad M(x) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{1+x^2}.$$

In Hardy's theorem, as in Ramanujan's work, both $L(x)$ and $M(x)$ are analytic for all real x and so, being even, are expressible as power series in x^2. This restriction involves an additional condition on $\chi(u)$. Here $L(x)$, if defined by evenness for negative x, is not analytic at the origin. Hence the above theorem is more general than Hardy's result.

Hardy states his conditions in terms of $\phi(u)$, where

$$\frac{\chi(u)}{2^{\frac{1}{2}u}\Gamma(\frac{1}{2}u+\frac{1}{2})} = \frac{\phi(u)}{\Gamma(u+1)};$$

and $\phi(u) = 0$ for positive odd integral u. Apart from this restriction, his result is equivalent to Theorem 1.

Another example is given by $\chi(u) \equiv 1$. In this case we obtain the self-reciprocal function

$$L(x) = M(x) = \sum_0^\infty \frac{(-1)^n}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+\frac{1}{2})}x^n,$$

which has been given by Hardy and Titchmarsh.* In the paper referred to, this function is given in the form

$$1-xe^{\frac{1}{2}x^2}\int\limits_x^\infty e^{-\frac{1}{2}t^2}\,dt = \int\limits_0^\infty e^{-\frac{1}{2}w^2-xw}w\,dw$$

which on expansion in powers of x yields the series obtained here.

* G. H. Hardy and E. C. Titchmarsh, 'Self-reciprocal functions': *Quart. J. of Math.* (Oxford), 1 (1930), 210, example 4.

3. THEOREM 2. *Let $p(u)$ be the reciprocal of an integral function taking real values on the real axis. Let $\chi(u)$ be an integral function satisfying the conditions*

$$\frac{\chi(\sigma+it)}{p(\sigma+1+it)} = O(e^{A\sigma+B|t|}), \qquad \frac{\chi(-\sigma+it)}{p(\sigma+it)} = O(e^{C\sigma+D|t|}),$$

where $B < \pi$, $D < \pi$, for $\sigma > 0$. Then

$$k(s) = \frac{p(s)}{p(1-s)}$$

satisfies the equations $k(s)k(1-s) = 1$, and $|k(\frac{1}{2}+it)| = 1$. If

$$\frac{K_1(x)}{x} = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{k(s)x^{-s}}{1-s}\,ds,$$

where the integral exists in the mean-square sense at infinity and K_1/x is $L^2(0,\infty)$, then

$$\sum_0^\infty \frac{(-1)^n\chi(n)}{p(n+1)}x^n$$

converges for $0 < x < e^{-A}$, and represents a function $L(x)$ regular for all positive x and of integrable square in $(0,\infty)$; and

$$\frac{d}{dy}\int_0^\infty \frac{K_1(xy)}{x} L(x)\,dx = M(y)$$

for $y > 0$, where

$$M(y) = \sum_0^\infty \frac{(-1)^n\chi(-n-1)}{p(n+1)}y^n$$

for $0 < y < e^{-C}$, and is the analytic continuation of the power-series, for all positive y. Also

$$\frac{d}{dy}\int_0^\infty \frac{K_1(xy)}{x} M(x)\,dx = L(y)$$

for $y > 0$. Thus $L(x)$ and $M(x)$ are Watson transforms with kernel K_1/x.

We have, by Cauchy's theorem,

$$L(x) = \sum_0^\infty \frac{(-1)^n\chi(n)}{p(n+1)}x^n = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi\chi(u)x^u\,du}{p(u+1)\sin\pi u}$$

if $-1 < c < 0$. As usual, the integral representation holds at first for $0 < x < e^{-A}$, and by analytic continuation for all positive x. It is plain that $L(x) = O(x^c)$ both for large and for small x, and hence taking in turn $c > \frac{1}{2}$ and $c < \frac{1}{2}$, we see that $L(x)$ is $L^2(0, \infty)$.

It follows that

$$\int_{\delta}^{\Delta} \frac{K_1(xy)}{x} L(x)\,dx = -\frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\pi\chi(u)\,du}{\sin \pi u\, p(u+1)} \int_{\delta}^{\Delta} x^u \frac{K_1(xy)}{x}\,dx$$

for $y > 0$. But, from the theory of the Mellin transform,

$$\int_{\delta}^{\Delta} x^u \frac{K_1(xy)}{x}\,dx,$$

as $\Delta \to \infty$ and $\delta \to 0$, converges in mean square to $-y^{-u}k(u+1)/u$, and also

$$\frac{\chi(u)}{\sin \pi u\, p(u+1)}$$

is $L^2(-\frac{1}{2}+i\infty, -\frac{1}{2}+i\infty)$. Hence, by the usual theory of mean convergence,

$$\int_{0}^{\infty} \frac{K_1(xy)}{x} L(x)\,dx = \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\pi\chi(u)y^{-u}k(u+1)\,du}{u \sin \pi u\, p(u+1)}$$

$$= \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\pi\chi(u)y^{-u}\,du}{u \sin \pi u\, p(-u)} = \sum_{1}^{\infty} \frac{(-1)^{n-1}\chi(-n)}{np(n)} y^n.$$

The final series converges for $0 < y < e^{-C}$, and as usual represents a function regular for all positive y. Hence

$$\frac{d}{dy} \int_{0}^{\infty} L(x) \frac{K_1(xy)}{x}\,dx = M(y)$$

for $y > 0$. The reciprocal relation can be obtained at once from the general theory of Watson transforms or by replacing $\chi(u)$ by $\chi(-u-1)$ in the relation already obtained. Also

$$\int_{0}^{\infty} \{L(x)\}^2\,dx = \int_{0}^{\infty} \{M(x)\}^2\,dx.$$

4. Examples. (i) Let $p(u) \equiv 1$. Then

$$L(x) = \sum_0^\infty (-1)^n \chi(n) x^n$$

and $K_1 = 0$ for $x < 0$, $K_1 = 1$ for $x > 1$, so that

$$\frac{d}{dy} \int_{1/y}^\infty \frac{L(x)}{x}\, dx = \frac{1}{y} L\left(\frac{1}{y}\right) = \sum_0^\infty (-1)^n \chi(-n-1) y^n.$$

The series $\sum (-1)^n \chi(n) y^{-n-1}$ converges for large y, and the series $\sum (-1)^n \chi(-n-1) y^n$ converges for small y. These series may not converge for any common values of y, but represent the same analytic function. Suppose, for example, that $\chi(u) \equiv 1$. Then

$$L(x) = M(x) = \frac{1}{1+x},$$

and this function is expansible as $\sum (-1)^n x^{-n-1}$ for $x > 1$, and as $\sum (-1)^n x^n$ for $x < 1$.

(ii) Taking $$p(u) = 2^{\frac{1}{2}(u-1)}\Gamma(\tfrac{1}{2}u),$$

$$L(x) = \sum_0^\infty \frac{(-1)^n \chi(n)}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+\frac{1}{2})} x^n, \qquad M(x) = \sum_0^\infty \frac{(-1)^n \chi(-n-1)}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+\frac{1}{2})} x^n$$

and $$\frac{d}{dy} \int_0^\infty L(x) \frac{\sin(xy)}{x}\, dx = M(y).$$

This is the integrated form of Theorem 1.

(iii) If $p(u) = \operatorname{cosec} \frac{1}{2}\pi u$,

$$K_1(x) = \frac{1}{\pi} \log \left| \frac{1+x}{1-x} \right|,$$

$$L(x) = \sum_0^\infty (-1)^n \chi(2n) x^{2n}, \qquad M(x) = \sum_0^\infty (-1)^n \chi(-2n-1) x^{2n},$$

and $$\frac{1}{\pi} \frac{d}{dy} \int_0^\infty \frac{L(x)}{x} \log \left| \frac{1+xy}{1-xy} \right| dx = M(y).$$

Formal differentiation under the integral sign yields

$$\frac{2}{\pi} \int_0^\infty \frac{L(x)}{1-x^2y^2}\, dx = M(y).$$

I do not attempt to justify this differentiation directly, but show

that the result is true by an application of the method used in the proof of Theorem 1.

5. THEOREM 3. *Let $\chi(u)$ be an integral function satisfying*

$$\chi(\sigma+it) = O(e^{A|\sigma|+B|t|})$$

with $B < \frac{1}{2}\pi$, for all σ and t. Then the series

$$\sum_0^\infty (-1)^n \chi(2n) x^{2n}$$

converges for $e^{-A} < x < e^{A}$, and represents a function $L(x)$ regular for all real x; and

$$\frac{2}{\pi}\int_0^\infty \frac{L(x)}{1-x^2y^2}\,dx = M(y),$$

where $$M(y) = \sum_0^\infty (-1)^n \chi(-2n-1) y^{2n}$$

for $-e^{-A} < y < e^{-A}$, and is regular for all real y. Also,

$$\frac{2}{\pi}\int_0^\infty \frac{M(x)}{1-x^2y^2}\,dx = L(y).$$

As in Theorem 1, we have

$$L(x) = \sum_0^\infty (-1)^n \chi(2n) x^{2n}$$

$$= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi\chi(u)\sin\frac{1}{2}\pi(u+1)x^u}{\sin\pi u}\,du$$

for $-1 < c < 0$, the series converging for $-e^{-A} < x < e^{-A}$, and the integral converging uniformly for $0 < \delta \leqslant x \leqslant \Delta < \infty$. Suppose now that y is fixed and $y > 0$. Then

$$\left(\int_\delta^{y^{-1}-\eta} + \int_{y^{-1}+\eta}^{\Delta}\right)\frac{L(x)\,dx}{1-x^2y^2}$$

$$= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi\chi(u)\sin\frac{1}{2}\pi(u+1)\,du}{\sin\pi u}\left(\int_\delta^{y^{-1}-\eta} + \int_{y^{-1}+\eta}^{\Delta}\right)\frac{x^u\,du}{1-x^2y^2}$$

for $y > 0$. Also, for small x,

$$\frac{x^u}{1-x^2y^2} = O(x^c)$$

and, for large x, $$\frac{x^u}{1-x^2y^2} = O(x^{c-2})$$

uniformly throughout the range of integration of u; and so it follows that we may replace δ by 0, and Δ by ∞ in the above relation, so that

$$\left(\int_0^{y^{-1}-\eta}+\int_{y^{-1}+\eta}^{\infty}\right)\frac{L(x)\,dx}{1-x^2y^2}$$

$$=-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi\chi(u)\sin\frac{1}{2}\pi(u+1)\,du}{\sin\pi u}\left(\int_0^{y^{-1}-\eta}+\int_{y^{-1}+\eta}^{\infty}\right)\frac{x^u\,dx}{1-x^2y^2}.$$

Also $$\left|\int_{y^{-1}-\eta}^{y^{-1}+\eta}\frac{x^u\,dx}{1-x^2y^2}\right|=|y^{-u-1}|\left|\int_{1-y\eta}^{1+y\eta}\frac{t^u\,dt}{1-t^2}\right|,$$

and, since $$\frac{t^u}{1-t^2}=\frac{1}{2(1-t)}+\frac{1}{2(1+t)}+\frac{t^u-1}{1-t^2},$$

$$\int_{1-y\eta}^{1+y\eta}\frac{t^u\,dt}{1-t^2}=\frac{1}{2}\int_{1-y\eta}^{1+y\eta}\frac{dt}{1+t}+\int_{1-y\eta}^{1+y\eta}\frac{t^u-1}{1-t^2}\,dt=\frac{1}{2}\log\frac{2+y\eta}{2-y\eta}+\int_{1-y\eta}^{1+y\eta}\frac{t^u-1}{1-t^2}\,dt.$$

Hence

$$\left|\int_{y^{-1}-\eta}^{y^{-1}+\eta}\frac{x^u\,dx}{1-x^2y^2}\right|\leqslant A\eta+B\int_{1-y\eta}^{1+y\eta}\left|\frac{t^u-1}{1-t}\right|dt<C\eta+D\eta|u|,$$

where A, B, C, D are independent of the imaginary part of u. Therefore

$$\int_{c-i\infty}^{c+i\infty}\frac{\chi(u)\sin\frac{1}{2}\pi(u+1)\,du}{\sin\pi u}\int_{y^{-1}-\eta}^{y^{-1}+\eta}\frac{x^u\,dx}{1-x^2y^2}$$

is convergent and tends to zero as $\eta\to 0$; and

$$\frac{2}{\pi}\int_0^{\infty}\frac{L(x)\,dx}{1-x^2y^2}=-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi\chi(u)\sin\frac{1}{2}\pi(u+1)\,du}{\sin\pi u}\,\frac{2}{\pi}\int_0^{\infty}\frac{x^u\,dx}{1-x^2y^2}$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi\chi(u)y^{-u-1}\,du}{2\sin\frac{1}{2}\pi(u+1)}=M(y)$$

for $0<y<e^{-A}$.

As before, since the condition on $\chi(u)$ is symmetrical about the

origin, we may replace $\chi(u)$ by $\chi(-u-1)$, and thereby obtain the reciprocal formula

$$\frac{2}{\pi}\int_0^\infty \frac{M(x)\,dx}{1-x^2y^2} = L(y).$$

Example. Let $\chi(u) \equiv 1$. Then

$$L(x) = M(x) = \frac{1}{1+x^2},$$

and we obtain a function self-reciprocal with respect to the kernel

$$\frac{2}{\pi}\frac{1}{1-x^2}.$$

Generally, any function self-reciprocal with respect to the kernel K_1/x, and obtainable from Theorem 3, will be of the form

$$-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi\chi(u)x^u\,du}{2\sin\frac{1}{2}\pi u} = \sum_0^\infty (-1)^n\chi(2n)x^{2n},$$

where $\chi(u)$ is an even function of $u+\frac{1}{2}$. For instance, if

$$\chi(u) = (u+\tfrac{1}{2})^{2n},$$

we obtain the sequence of self-reciprocal functions

$$\left(x\frac{d}{dx}+\frac{1}{2}\right)^{2n}\frac{1}{1+x^2}.$$

HYPERGEOMETRIC PARTIAL DIFFERENTIAL EQUATIONS (III)

By T. W. CHAUNDY (*Oxford*)

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1. The equation $y^m f(\delta)V = x^m g(\delta')V$

In two previous papers under this title (**2, 3**) I have discussed the solution of those partial differential equations of the form

$$f(\delta)F(\delta')V = xy\,g(\delta)\,G(\delta')V \quad \left(\delta, \delta' \equiv x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right) \tag{1}$$

in which the operators of 'dominant' order are found on the same side of the equation: that is to say, if p, q, P, Q are the degrees of f, g, F, G respectively, then

$$p \geqslant q, \qquad P \geqslant Q, \tag{2}$$

(or, of course, *both* inequalities might be reversed). I now consider certain equations of the same form (1) under the opposite conditions

$$p > q, \qquad P < Q. \tag{3}$$

I had previously proposed to distinguish these two classes of equation as *hyperbolic* and *parabolic*, but I have since realized that this would be a misleading extension of accepted terms. The real distinction (as I see it) arises in this way. Equations of the first type have for their characteristics the two families of axi-parallels $x =$ constant, $y =$ constant, i.e. the characteristics are always equal in number to the independent variables. Equations of the second type fall into two classes. If the aggregate orders of the operators are the same on both sides of the equation, i.e. if $p+P = q+Q$ in (3), then the equation will have other characteristics instead of (or in addition to) the axi-parallels and so, in general, will have more characteristics than there are independent variables: the characteristics will be 'excessive' as I have elsewhere proposed to say. Thus the equation

$$\delta(\delta-1)(\delta-2)V = x^3y^{-3}\delta'(\delta'-1)(\delta'-2)V, \quad \text{i.e. } \frac{\partial^3 V}{\partial x^3} = \frac{\partial^3 V}{\partial y^3},$$

has the three families of characteristics

$$x+\omega y = \text{constant} \quad (\omega^3 = 1).$$

On the other hand, in the equation

$$\delta(\delta-1)V = x^2y^{-1}\delta' V, \quad \text{i.e. } \frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial y},$$

where the orders differ on the two sides, the characteristics are the single set of axi-parallels $y = \text{constant}$, i.e. the characteristics are 'deficient', and the equation is *parabolic* in the proper sense of the term.

In this paper I shall still exclude the properly parabolic case and consider only the equations in which $p+Q = P+q$, when (as I have said) the characteristics are, in general, excessive. An equation evidently of this type is*

$$f(\delta)V = x^a y^b g(\delta')V \tag{4}$$

in which f, g are polynomials of *equal* degree m. By a suitable transformation $x' = x^h$, $y' = y^k$ we can reduce it to the simpler form

$$y^m f(\delta)V = x^m g(\delta')V, \tag{5}$$

in which we may further suppose that the leading coefficients in f, g are unity. In this case the terms of highest order will be

$$y^m\,\delta(\delta-1)...(\delta-m+1)V - x^m\,\delta'(\delta'-1)...(\delta'-m+1)V,$$

i.e.

$$x^m y^m\left(\frac{\partial^m V}{\partial x^m} - \frac{\partial^m V}{\partial y^m}\right),$$

and the characteristics are therefore the m pencils of straight lines

$$x+\omega y = \text{constant} \quad (\omega^m = 1).$$

I shall use a technique similar to that with which I have previously discussed a class of equations with 'excessive' characteristics.†

2. The solution of Cauchy's problem

As usual we suppose given an arbitrary point P with coordinates (X, Y) and an arbitrary base-curve. The solution of 'Cauchy's problem' consists in determining the value $V(X, Y)$ at P of any solution $V(x, y)$ of the given equation in terms of its value and that of certain of its derivatives along the given base-curve.

Draw through P the pencil of the m characteristics

$$x-X+\omega_r(y-Y) = 0 \quad (r = 1,..., m) \tag{6}$$

* This, of course, does not represent the *most general* equation of the type. The general equation, with operators in δ and δ' on both sides, will have axi-parallel characteristics in addition to those of the type $x+\omega y = \text{constant}$.

† See (**4**).

cutting the base-curve in the m points $A_1,\ldots, A_m$, and take any convenient origin O on the base-curve and any convenient curve OP connecting O to P.

I then postulate a set of m 'fundamental solutions' (or 'Green's functions')

$$U_r(X,Y;x,y) \quad (r=1,\ldots, m) \tag{7}$$

each associated with a characteristic PA_r and having the following properties:

(A) *they are solutions, in variables* (X,Y), *of the given equation* (5) *and, in variables* (x,y), *of its adjoint equation;*

(B) *they satisfy the 'null condition'*

$$\sum_{r=1}^{m} U_r = 0;$$

(C) *along the characteristic* PA_r *the associated solution* U_r *vanishes with all its derivatives in* x, y *of order less than* $m-2$: *its derivatives of order* $m-2$ *have the values*

$$\frac{\partial^{m-2}U_r}{\partial x^{m-2-q}\partial y^q} = \frac{\omega_r^{q+1}}{m}\left(\frac{x}{X}\right)^a\left(\frac{y}{Y}\right)^b (xyXY)^{-\frac{1}{2}(m-1)}$$

along this characteristic, ma, mb *being the coefficients of the terms of order* $m-1$ *in* $f(\delta)$, $g(\delta')$.

Observe in (C) that, since every derivative of U_r of order $m-3$ vanishes along PA_r, we necessarily have, on differentiating along this line,

$$\omega_r\frac{\partial^{m-2}U_r}{\partial x^{m-2-q}\partial y^q} - \frac{\partial^{m-2}U_r}{\partial x^{m-3-q}\partial y^{q+1}} = 0,$$

which gives

$$\frac{\partial^{m-2}U_r}{\partial x^{m-2-q}\partial y^q} = \omega_r^q\,\theta_r \tag{8}$$

for some θ_r; the second part of the condition (C) serves merely to fix θ_r, defining it as

$$\theta_r = \frac{\omega_r}{m}\left(\frac{x}{X}\right)^a\left(\frac{y}{Y}\right)^b (xyXY)^{-\frac{1}{2}(m-1)} \tag{9}$$

I now make the convention that λ, λ' denote the operators δ, δ' acting only on V and its derivatives, and $-\mu$, $-\mu'$ the same operators acting only on the U_r and their derivatives, so that, on mixed operands in U_r, V, the full operators are

$$\delta = \lambda-\mu, \qquad \delta' = \lambda'-\mu'.$$

Thus, if V is any solution of the given equation (5),

$$[y^m f(\lambda)-x^m g(\lambda')]U_r V = 0; \tag{10}$$

and, since, by (A), every U_r is a solution of the adjoint equation

$$[y^m f(-\delta)-x^m g(-\delta)]U = 0,$$

we have also
$$[y^m f(\mu)-x^m g(\mu')]U_r V = 0. \tag{11}$$

By (10), (11)

$$\left[(\lambda-\mu)y^m\frac{f(\lambda)-f(\mu)}{\lambda-\mu}-(\lambda'-\mu')\,x^m\frac{g(\lambda')-g(\mu')}{\lambda'-\mu'}\right]U_r V = 0,$$

i.e.
$$\Theta \equiv \delta\left[y^m\frac{f(\lambda)-f(\mu)}{\lambda-\mu}U_r V\right]-\delta'\left[x^m\frac{g(\lambda')-g(\mu')}{\lambda'-\mu'}U_r V\right] = 0.$$

Thus over any closed area

$$\iint \Theta\frac{dxdy}{xy} = 0,$$

and so, by the Green-Stokes theorem (in δ, δ' as fundamental operators),

$$\int\left\{\left(x^m\frac{g(\lambda')-g(\mu')}{\lambda'-\mu'}U_r V\right)\frac{dx}{x}+\left(y^m\frac{f(\lambda)-f(\mu)}{\lambda-\mu}U_r V\right)\frac{dy}{y}\right\} = 0$$

round the boundary of this area, i.e. round any closed curve.

Take the closed curve to be the curve $OA_r PO$ made up of the arc OA_r of the base-curve, the segment $A_r P$ of the characteristic corresponding to U_r, and the arbitrary arc PO; and sum in r. Then, along the arc PO, which is common to all the m integrals, U_r appears as $\sum U_r$, which vanishes in virtue of the null condition (B). Thus the part of the summed integral corresponding to the arbitrary curve PO vanishes, and we have left

$$\sum_{r=1}^{m}\left[\int_{A_r}^{P}+\int_{O}^{A_r}\left\{\left(x^m\frac{g(\lambda')-g(\mu')}{\lambda'-\mu'}U_r V\right)\frac{dx}{x}+\left(y^m\frac{f(\lambda)-f(\mu)}{\lambda-\mu}U_r V\right)\right\}\frac{dy}{y}\right] = 0. \tag{12}$$

The second integral in the summation is taken always along the base-curve, i.e. in terms of Cauchy's problem, it involves only 'known' values of the unknown V and its derivatives (apart from the functions U_r supposed known). Thus we need consider only the reduction of the first integral

$$\int_{A_r}^{P}\left\{\left(x^m\frac{g(\lambda')-g(\mu')}{\lambda'-\mu'}U_r V\right)\frac{dx}{x}+\left(y^m\frac{f(\lambda)-f(\mu)}{\lambda-\mu}U_r V\right)\frac{dy}{y}\right\} \tag{13}$$

taken along the characteristic $A_r P$.

3. Reduction of the integral along the characteristic

In considering the reduction of (13) I shall, for brevity, omit the suffix r from the associated A_r, U_r, ω_r, θ_r, restoring it when I come to substitute the reduced value in the summation (12). Now, in the first place, in view of (C), we need retain along this characteristic no derivatives of U of order lower than $m-2$, i.e. we retain only powers of μ, μ' of orders $m-1$, $m-2$. To this order we have

$$\frac{f(\mu)-f(\lambda)}{\mu-\lambda}UV = (\mu^{m-1}+\mu^{m-2}\lambda+ma\,\mu^{m-2})UV$$
$$= V[(-\delta)^{m-1}+ma(-\delta)^{m-2}]U+(\delta V)(-\delta)^{m-2}U,$$

and so for g. Again, sufficiently to this order,

$$\delta^{m-1} = \delta(\delta-1)...(\delta-m+2)+\tfrac{1}{2}(m-1)(m-2)\delta(\delta-1)...(\delta-m+3)$$
$$= x^{m-1}\frac{\partial^{m-1}}{\partial x^{m-1}}+\tfrac{1}{2}(m-1)(m-2)x^{m-2}\frac{\partial^{m-2}}{\partial x^{m-2}},$$

and so for δ'. Thus, along PA, the integrand of (13) is effectively (after rearrangement)

$$(-xy)^{m-1}\left\{V\left[\frac{\partial^{m-1}U}{\partial x^{m-1}}dy+\frac{\partial^{m-1}U}{\partial y^{m-1}}dx+\frac{\tfrac{1}{2}(m-1)(m-2)-ma}{x}\frac{\partial^{m-2}U}{\partial x^{m-2}}dy+\right.\right.$$
$$\left.\left.+\frac{\tfrac{1}{2}(m-1)(m-2)-mb}{y}\frac{\partial^{m-2}U}{\partial y^{m-2}}dx\right]-\frac{\partial V}{\partial x}\frac{\partial^{m-2}U}{\partial x^{m-2}}dy-\frac{\partial V}{\partial y}\frac{\partial^{m-2}U}{\partial y^{m-2}}dx\right\}. \tag{14}$$

Now differentiation along PA is effected by the operator $\omega\partial/\partial x-\partial/\partial y$. Thus differentiating $\omega^{m-1}\theta$ along PA we get, by (8),

$$\omega^{m-q}\frac{\partial^{m-1}U}{\partial x^{m-q-1}\partial y^{q}}-\omega^{m-q-1}\frac{\partial^{m-1}U}{\partial x^{m-q-2}\partial y^{q+1}} = \omega^{m-1}\left(\omega\frac{\partial\theta}{\partial x}-\frac{\partial\theta}{\partial y}\right).$$

Summing this in q from 0 to $m-2$ we have, since $\omega^m = 1$,

$$\frac{\partial^{m-1}U}{\partial x^{m-1}}-\omega\frac{\partial^{m-1}U}{\partial y^{m-1}} = (m-1)\left(\frac{\partial\theta}{\partial x}-\omega^{-1}\frac{\partial\theta}{\partial y}\right). \tag{15}$$

Then, in (14), taking the first pair of terms in the coefficient of V we have

$$\frac{\partial^{m-1}U}{\partial x^{m-1}}dy+\frac{\partial^{m-1}U}{\partial y^{m-1}}dx$$
$$= \left(\frac{\partial^{m-1}U}{\partial x^{m-1}}-\omega\frac{\partial^{m-1}U}{\partial y^{m-1}}\right)dy, \quad \text{since } dx+\omega\,dy = 0 \text{ along } AP,$$
$$= (m-1)\left(\frac{\partial\theta}{\partial x}-\omega^{-1}\frac{\partial\theta}{\partial y}\right)dy, \quad \text{by (15).}$$

The second pair of terms in the coefficient of V can be written

$$\left\{\frac{\frac{1}{2}(m-1)(m-2)-ma}{x}-\frac{\frac{1}{2}(m-1)(m-2)-mb}{\omega y}\right\}\theta\,dy$$

$$=-m\left(\frac{\partial\theta}{\partial x}-\omega^{-1}\frac{\partial\theta}{\partial y}\right)dy-(m-1)\left(\frac{\theta}{x}-\frac{\theta}{\omega y}\right)dy,$$

since, by (9),

$$\frac{\partial\theta}{\partial x}-\omega^{-1}\frac{\partial\theta}{\partial y}=\left\{\frac{a-\frac{1}{2}(m-1)}{x}-\frac{b-\frac{1}{2}(m-1)}{\omega y}\right\}\theta.$$

Thus the coefficient of V in (14) reduces to

$$-\left(\frac{\partial\theta}{\partial x}-\omega^{-1}\frac{\partial\theta}{\partial y}\right)dy-(m-1)\left(\frac{\theta}{x}-\frac{\theta}{\omega y}\right)dy.$$

The whole expression (14) can therefore be written

$$\frac{(-xy)^{m-1}}{\omega}\left\{-V\left(\omega\frac{\partial\theta}{\partial x}-\frac{\partial\theta}{\partial y}\right)dy-(m-1)V\left(\frac{\omega}{x}-\frac{1}{y}\right)\theta\,dy-\theta\left(\omega\frac{\partial V}{\partial x}-\frac{\partial V}{\partial y}\right)dy\right\}$$

$$=\frac{(-xy)^{m-1}}{\omega}\left\{V\left(\frac{\partial\theta}{\partial x}dx+\frac{\partial\theta}{\partial y}dy\right)+(m-1)\theta V\left(\frac{dx}{x}+\frac{dy}{y}\right)+\right.$$

$$\left.+\theta\left(\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy\right)\right\}$$

$$=\frac{(-1)^{m-1}}{\omega}d(x^{m-1}y^{m-1}\theta V).$$

The integrand (14) is thus an exact derivative along PA, and so the integral (13) reduces (with restored suffixes) to

$$\frac{(-1)^{m-1}}{\omega_r}[x^{m-1}y^{m-1}\theta_r V]_{A_r}^{P}$$

$$=\frac{(-1)^{m-1}}{m}\left[\left(\frac{x}{X}\right)^{a+\frac{1}{2}(m-1)}\left(\frac{y}{Y}\right)^{b+\frac{1}{2}(m-1)}V\right]_{A_r}^{P},\quad\text{by (9)},$$

$$=\frac{(-1)^{m-1}}{m}\left\{V(X,Y)-\left[\left(\frac{x}{X}\right)^{a+\frac{1}{2}(m-1)}\left(\frac{y}{Y}\right)^{b+\frac{1}{2}(m-1)}V\right]^{A_r}\right\}.$$

Hence finally from (12) we get

$$V(X,Y)$$

$$=(-)^m\sum_{r=1}^{m}\int_{O}^{A_r}\left\{\left[x^m\frac{g(\lambda')-g(\mu')}{\lambda'-\mu'}U_r V\right]\frac{dx}{x}+\left[y^m\frac{f(\lambda)-f(\mu)}{\lambda-\mu}U_r V\right]\frac{dy}{y}\right\}+$$

$$+\frac{1}{m}\sum_{r=1}^{m}\left[\left(\frac{x}{X}\right)^{a+\frac{1}{2}(m-1)}\left(\frac{y}{Y}\right)^{b+\frac{1}{2}(m-1)}V\right]^{A_r}.\tag{16}$$

The terms to be evaluated at A_r can be replaced, if it is preferred, by

$$\frac{1}{m}\left[\sum_{r=1}^{m}\int\limits_{O}^{A_r}\left(\frac{x}{X}\right)^{a+\frac{1}{2}(m-1)}\left(\frac{y}{Y}\right)^{b+\frac{1}{2}(m-1)}\left\{\left(\frac{\partial V}{\partial x}+\frac{a+\frac{1}{2}(m-1)}{x}V\right)dx+\right.\right.$$
$$\left.\left.+\left(\frac{\partial V}{\partial y}+\frac{b+\frac{1}{2}(m-1)}{y}V\right)dy\right\}+\left(\frac{x}{X}\right)^{a+\frac{1}{2}(m-1)}\left(\frac{y}{Y}\right)^{b+\frac{1}{2}(m-1)}V\right]^{O},$$

so that the whole expression on the right of (16) is then composed of a sum of integrals from O to A_r together with a term at O.

In either form (16) provides the solution of Cauchy's problem,* for it gives the value of any solution V at the arbitrary point (X, Y) in terms of the values of V and its derivatives along the given base-curve, so long as we know the set of fundamental solutions

$$U_r(X, Y; x, y).$$

To the problem of determining this fundamental set we now turn.

4. The fundamental solutions

To avoid excess of algebraic detail I take the third-order equation as sufficiently illustrative. As a further simplification I note that the substitution

$$V = x^{-h}y^{-k}V' \tag{17 A}$$

transforms the given equation (5) into the form

$$\{y^m f(\delta-h)-x^m g(\delta'-k)\}V' = 0,$$

of which the adjoint is

$$\{y^m f(-\delta-h)-x^m g(-\delta'-k)\}U' = 0.$$

Hence the substitution

$$U_r = x^h y^k X^{-h} Y^{-k} U_r' \tag{17 B}$$

simultaneously applied to a set of fundamental solutions of (5) preserves the property (A) of § 2. The null condition is evidently invariant for such a substitution, and, lastly, the terms of order $m-1$ in $f(\delta-h)$, $g(\delta'-k)$ have coefficients $m(a-h)$, $m(b-k)$, and so the

* Or, more strictly, *a possible solution* of Cauchy's problem. For, if $m > 2$, not all the characteristics are real. Thus some A_r will be complex and the integrals will be, fundamentally, integrals in a complex plane, their values possibly depending on the path of integration. But the argument appears to show that (16) remains a solution for all such values. The underlying problem is then how far in such a case Cauchy's problem is 'correctly set', i.e. how far the data are adequate to define a solution. That is a problem I do not attempt to discuss.

substitution also preserves the property (C). Thus a set of fundamental solutions remains a set of fundamental solutions.

I use the substitution to remove the absolute terms in f, g, so that the differential equation may be considered in the form

$$y^3\delta(\delta+3p_1+1)(\delta+3p_2+2)V = x^3\delta'(\delta'+3q_1+1)(\delta'+3q_2+2)V, \tag{18}$$

and I begin by considering the case in which p_1, p_2, q_1, q_2 are all *positive integers*.* The adjoint of (18) is

$$y^3\,\delta(\delta-3p_1-1)(\delta-3p_2-2)U = x^3\,\delta'(\delta'-3q_1-1)(\delta'-3q_2-2)U. \tag{19}$$

To obtain its solution I introduce the operators

$$\left.\begin{aligned} \Delta_x^1(p_1) &\equiv (\delta-1)(\delta-4)...(\delta-3p_1+2) \\ \Delta_x^2(p_2) &\equiv (\delta-2)(\delta-5)...(\delta-3p_2+1) \\ \Delta_y^1(q_1) &\equiv (\delta'-1)(\delta'-4)...(\delta'-3q_1+2) \\ \Delta_y^2(q_2) &\equiv (\delta'-2)(\delta'-5)...(\delta'-3q_2+1) \end{aligned}\right\}, \tag{20}$$

the factors in each product descending by differences of 3, and substitute in (19)

$$U = \Delta_x^1(p_1)\Delta_x^2(p_2)\Delta_y^1(q_1)\Delta_y^2(q_2)W. \tag{21}$$

The equation can then be rewritten

$$(\delta-4)...(\delta-3p_1-1).(\delta-5)...(\delta-3p_2-2).(\delta'-4)...(\delta'-3q_1-1)\times$$
$$\times(\delta'-5)...(\delta'-3q_2-2)[y^3\,\delta(\delta-1)(\delta-2)-x^3\,\delta'(\delta'-1)(\delta'-2)]W = 0$$

and is therefore satisfied, if W is any solution of the differential equation

$$[y^3\,\delta(\delta-1)(\delta-2)-x^3\,\delta'(\delta'-1)(\delta'-2)]W = 0,$$

i.e. of the equation $$\frac{\partial^3 W}{\partial x^3}-\frac{\partial^3 W}{\partial y^3} = 0.$$

Thus, from (21), a possible solution of (19) is

$$U = \Delta_x^1(p_1)\Delta_x^2(p_2)\,\Delta_y^1(q_1)\Delta_y^2(q_2)\,\phi(x+\omega y), \tag{22}$$

where $\omega^3 = 1$ and the function ϕ is arbitrary.

We now need to distinguish the signs of p_1-p_2 and q_1-q_2. It is sufficiently illustrative if we suppose

$$p_2 \geqslant p_1, \qquad q_1 \geqslant q_2+1. \tag{23}$$

Substituting $$V = x^{-(3p_2+2)}y^{-(3q_1+1)}V' \tag{24}$$

* The partial differential equation has then an analogy with the ordinary differential equation discussed in (**1**).

in (18) we can write that equation as

$$y^3\,\delta\{\delta-3(p_2-p_1)-1\}\{\delta-3p_2-2\}V' \\ = x^3\,\delta\{\delta-3q_1-1\}\{\delta-3(q_1-q_2-1)-2\}V'.$$

Since p_2-p_1, q_1-q_2-1 are positive, as well as p_2, q_1, in virtue of the assumption (23), this equation is similar to (19) with changed parameters. Thus, by (22), it has a solution of the form

$$V' = \Delta_x^1(p_2-p_1)\Delta_x^2(p_2)\,\Delta_y^1(q_1)\Delta_y^2(q_1-q_2-1)\,\psi(x+\omega y). \tag{25}$$

We transform this to variables X, Y in accordance with the terms of (A),* and then combine (22), (24), (25) in the form

$$U_\omega = \tfrac{1}{3}X^{-(3p_2+2)}(\omega Y)^{-(3q_1+1)}\Delta_x^1(p_1)\Delta_x^2(p_2)\,\Delta_y^1(q_1)\Delta_y^2(q_2)\times \\ \times\,\Delta_X^1(p_2-p_1)\Delta_X^2(p_2)\,\Delta_Y^1(q_1)\Delta_Y^2(q_1-q_2-1)\frac{\{(x-X)+\omega(y-Y)\}^{3p_2+3q_1}}{(3p_2+3q_1)!}. \tag{26}$$

This, then, is a solution, in variables x, y, of (19), and, in variables X, Y, of (18), and so condition (A) is satisfied.

5. Behaviour along a characteristic: the null condition

Now along the characteristic

$$(x-X)+\omega(y-Y) = 0 \tag{27}$$

the operand $\{(x-X)+\omega(y-Y)\}^{3p_2+3q_1}$ and its derivatives of order less than $3p_2+3q_1$ all vanish. The order of any operator $\Delta(h)$ is given by its parameter h, and so the eight operators in (26) have an aggregate order $3p_2+3q_1-1$. Thus U_ω vanishes along its associated characteristic (27).

In δU_ω the aggregate order of the operators equals the degree of the operand. Thus, along the associated characteristic, δU_ω has a value which is not zero and which can be found by retaining only the operators of highest order. This gives along the characteristic†

$$\delta U_\omega = \tfrac{1}{3}X^{-(3p_2+2)}(\omega Y)^{-(3q_1+1)}x^{p_1+p_2+1}(\omega y)^{q_1+q_2}X^{2p_2+p_1}(\omega Y)^{2q_1-q_2-1},$$

i.e.
$$\frac{\partial U_\omega}{\partial x} = \tfrac{1}{3}\omega\, x^{p_1+p_2}y^{q_1+q_2}X^{-(p_1+p_2+2)}Y^{-(q_1+q_2+2)}, \tag{28}$$

* The transformation requires us to define Δ_X, Δ_Y with such a convention of sign that

$$\Delta_X^1(p) \equiv (-\delta_X+1)(-\delta_X+4)\dots(-\delta_X+3p-2) \\ = (-)^p(\delta_X-1)(\delta_X-4)\dots(\delta_X-3p+2),$$

and so on.

† We must not overlook the conventions of sign inherent in Δ_X, Δ_Y.

which is the form required by (C), since, in (18),

$$a = p_1+p_2+1, \qquad b = q_1+q_2+1, \qquad \tfrac{1}{2}(m-1) = 1;$$

so for $\partial U_\omega/\partial y$.

To prove the null condition (B) we observe that U_ω can be expanded in the form $A+B\omega+C\omega^2$. Since B, C disappear in the summation $\sum U$, we have to show that the term in U_ω independent of ω is absent, i.e., from (26), that the coefficient of ω is zero in

$$\Delta_x^1(p_1)\Delta_x^2(p_2)\,\Delta_y^1(q_1)\Delta_y^2(q_2)\times$$
$$\times\Delta_X^1(p_2-p_1)\Delta_X^2(p_2)\,\Delta_Y^1(q_1)\Delta_Y^2(q_1-q_2-1)\{(x-X)+\omega(y-Y)\}^{3p_2+3q_1}.$$

Now the coefficient of ω in the expansion of $\{(x-X)+\omega(y-Y)\}^{3p_2+3q_1}$ is composed of terms of the form

$$(x-X)^{3h-1}(y-Y)^{3k+1} \quad (h+k = p_2+q_1)$$

(of course, with appropriate coefficients, which I now ignore). The terms in the expansion of $(x-X)^{3h-1}$ have one of the three forms

$$\text{(i) } x^{3a}X^{3b-1}, \qquad \text{(ii) } x^{3a-1}X^{3b}, \qquad \text{(iii) } x^{3a-2}X^{3b+1},$$

where $a+b = h$. Of these (i) is annihilated by $\Delta_X^2(p_2)$ unless $b \geqslant p_2+1$; (ii) by $\Delta_x^2(p_2)$ unless $a \geqslant p_2+1$; and (iii) by $\Delta_x^1(p_1)$ unless $a \geqslant p_1+1$, and by $\Delta_X^1(p_2-p_1)$ unless $b \geqslant p_2-p_1$. Thus no term survives in $(x-X)^{3h-1}$ unless, in every case, $a+b \geqslant p_2+1$, i.e. unless $h \geqslant p_2+1$.

Again, the expansion of $(y-Y)^{3k+1}$ consists of terms of one of the three forms

$$\text{(iv) } y^{3a}Y^{3b+1}, \qquad \text{(v) } y^{3a+1}Y^{3b}, \qquad \text{(vi) } y^{3a+2}Y^{3b-1},$$

where $a+b = k$. Of these (iv) is annihilated by $\Delta_Y(q_1)$ unless $b \geqslant q_1$; (v) by $\Delta_y(q_1)$ unless $a \geqslant q_1$; and (vi) by $\Delta_y(q)$ unless $a \geqslant q_2$, and by $\Delta_Y(q_1-q_2-1)$ unless $b \geqslant q_1-q_2$. Thus no term survives in $(y-Y)^{3k+1}$ unless, in every case, $a+b \geqslant q_1$, i.e. unless $k \geqslant q_1$.

Hence one or other binomial in the product $(x-X)^{3h-1}(y-Y)^{3k+1}$ is annihilated by the product of the Δ operators unless *simultaneously* $h \geqslant p_2+1$ and $k \geqslant q_1$. But this is impossible since $h+k = p_2+q_1$. Thus the functions (26) satisfy the null condition (B) as well as the other conditions (A), (C) predicated of a set of fundamental solutions.

This completes the solution of Cauchy's problem for the third-order equation (18) in the case when p_1, p_2, q_1, q_2 are positive integers.*

* Or, sufficiently, when they are integers of either sign: a transformation of the type (17) reduces the more general case to the case we have considered.

This is the case in which the fundamental solutions are polynomials. In preparation for the more general case in which they are transcendental functions I put the expression (26) into the form of a contour integral.

6. The fundamental solutions as contour integrals

Since $\{(x-X)+\omega(y-Y)\}^{3p_2+3q_1}/(3p_2+3q_1)!$ is the coefficient of $t^{3p_2+3q_1}$ in the expansion of $\exp\{(x-X)t+\omega(y-Y)t\}$, we can write the fundamental solution (26) as the contour integral

$$U_\omega = \frac{1}{6\pi i}X^{-(3p_2+2)}(\omega Y)^{-(3q_1+1)}\int^{(0+)} \Delta_x^1(p_1)\Delta_x^2(p_2)\Delta_y^1(q_1)\Delta_y^2(q_2)\times$$
$$\times\Delta_X^1(p_2-p_1)\Delta_X^2(p_2)\Delta_Y^1(q_1)\Delta_Y^2(q_1-q_2-1)\times$$
$$\times\exp\{(x-X)t+\omega(y-Y)t\}\frac{dt}{t^{3p_2+3q_1+1}},$$

which I rearrange as

$$U_\omega = \frac{1}{6\pi i}\int^{(0+)} \Delta_x^1(p_1)\Delta_x^2(p_2)e^{xt}.\Delta_y^1(q_1)\Delta_y^2(q_2)e^{\omega yt}\times$$
$$\times(Xt)^{-(3p_2+2)}\Delta_X^1(p_2-p_1)\Delta_X^2(p_2)e^{-Xt}\times$$
$$\times(\omega Yt)^{-(3q_1+1)}\Delta_Y^1(q_1)\Delta_Y^2(q_1-q_2-1)e^{-\omega Yt}.t^2\,dt.$$

Let us write this

$$U_\omega = \frac{1}{6\pi i}\int^{(0+)} \xi(xt)\eta(\omega yt)\bar{\xi}(Xt)\bar{\eta}(\omega Yt)t^2\,dt, \tag{29}$$

where
$$\xi(xt) \equiv \Delta_x^1(p_1)\Delta_x^2(p_2)e^{xt},$$
$$\bar{\xi}(Xt) \equiv (Xt)^{-(3p_2+2)}\Delta_X^1(p_2-p_1)\Delta_X^2(p_2)e^{-Xt}, \quad \text{etc.}$$

Thus $\xi(x) = \Delta_x^1(p_1)\Delta_x^2(p_2)e^x$, since the operators Δ_x involve only δ; and so for the other δ operators.

Consider the ordinary differential equation

$$\delta(\delta-3p_1-1)(\delta-3p_2-2)u = x^3u \tag{30}$$

formed by extracting those parts of (19) which involve x only, and write

$$u = \Delta_x^1(p_1)\Delta_x^2(p_2)w.$$

This substitution (in x) corresponds to the substitution (21) in x, y, and we similarly find that u satisfies (30) if w is any solution of

$$\delta(\delta-1)(\delta-2)w = x^3w,$$

i.e. of

$$\frac{d^3w}{dx^3} = w. \tag{31}$$

Taking the solution $w = e^x$, we get for the equation (30) the solution

$$u = \Delta_x^1(p_1)\Delta_x^2(p_2)e^x,$$

which is exactly $\xi(x)$. We similarly find that $\eta(y)$ is a solution of the ordinary equation

$$\delta'(\delta'-3q_1-1)(\delta'-3q_2-2)u' = y^3u', \tag{32}$$

extracted from (19) by suppressing x, δ; and that $\bar{\xi}(x)$, $\bar{\eta}(y)$ are solutions of the equations adjoint, in the operators δ, δ', to (30), (32), namely

$$\left.\begin{aligned} \delta(\delta+3p_1+1)(\delta+3p_2+2)v &= (-x)^3v \\ \delta'(\delta'+3q_1+1)(\delta'+3q_2+2)v &= (-y)^3v \end{aligned}\right\}, \tag{33}$$

which can, of course, be formed out of the original equation (18) adjoint to (19).

Conversely, with ξ, η, $\bar{\xi}$, $\bar{\eta}$ so defined as solutions of their respective differential equations, we see readily that (29) satisfies the conditions (A) postulated for the fundamental solutions and that this is true even when p_1, p_2, q_1, q_2 are no longer positive integers. More generally, if $\xi(x)$, $\eta(y)$ are solutions of the ordinary equations

$$f(-\delta)\xi = x^m\xi, \qquad g(-\delta')\eta = y^m\eta, \tag{34}$$

where f, g are polynomials of equal degree m, and $\bar{\xi}(x)$, $\bar{\eta}(y)$ solutions of their adjoint equations

$$f(\delta)\bar{\xi} = x^m\bar{\xi}, \qquad g(\delta')\bar{\eta} = y^m\bar{\eta}, \tag{35}$$

then it is clear that

$$\int^{(0+)} \xi(xt)\eta(\omega yt)\bar{\xi}(Xt)\bar{\eta}(\omega Yt)t^{m-1}\,dt \tag{36}$$

satisfies the conditions (A) for fundamental solutions of the partial differential equation (5). This is analogous to what has been proved in (3) for equations of the form (1).

7. The null condition for the contour-integral solutions

So far, then, to satisfy the condition (A), it has been sufficient that ξ, η, $\bar{\xi}$, $\bar{\eta}$ should be *any* solutions of the differential equations (34) and their adjoints. To satisfy the null condition (B) it is necessary to be more precise and to link suitably the η, $\bar{\eta}$ that appear in the set of fundamental solutions.

Returning to third-order equations, let us replace the particular solutions $\eta(y)$, $\eta(\omega y)$, $\eta(\omega^2y)$ of (32) by any linearly independent set

η_1, η_2, η_3. With these we associate, as is usually the case, the particular adjoint solutions* $\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3$ obtained by dividing the cofactors

$$\left\|\begin{matrix} \delta\eta_1 & \delta\eta_2 & \delta\eta_3 \\ \eta_1 & \eta_2 & \eta_3 \end{matrix}\right\|$$

of the Wronskian

$$W \equiv \begin{vmatrix} \delta^2\eta_1 & \delta^2\eta_2 & \delta^2\eta_3 \\ \delta\eta_1 & \delta\eta_2 & \delta\eta_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix}$$

by the Wronskian itself. Then the sum $\sum \eta(\omega yt)\bar{\eta}(\omega Yt)$ which appears in $\sum U_\omega$ will be replaced by the expression

$$W^{-1}\begin{vmatrix} \eta_1(yt) & \eta_2(yt) & \eta_3(yt) \\ \delta\eta_1(Yt) & \delta\eta_2(Yt) & \delta\eta_3(Yt) \\ \eta_1(Yt) & \eta_2(Yt) & \eta_3(Yt) \end{vmatrix}, \tag{37}$$

which is evidently invariant when η_1, η_2, η_3 are replaced by any linear combination of them (of course, with *constant* coefficients).

In other words (37) is *independent of the particular set of solutions* η_1, η_2, η_3, and we may therefore replace them by any other set without affecting $\sum U_\omega$. Now the theory of solving hypergeometric ordinary differential equations in power-series shows us that the equation (32) has, in general,† three solutions of the form

$$P_1(y^3), \qquad y^{3q_1+1}P_2(y^3), \qquad y^{3q_2+2}P_3(y^3),$$

where each $P(y^3)$ denotes a power-series of the form

$$a_0+a_1y^3+a_2y^6+\dots,$$

i.e. a power-series composed of non-negative integer powers of y^3. If we substitute these three solutions for η_1, η_2, η_3 in (37), we see that, regarded as a function of t, it is also of the form $P(t^3)$. Hence now $\sum U_\omega$ has the form

$$\frac{1}{6\pi i}\int^{(0+)} \xi(xt)\bar{\xi}(Xt)P(t^3)t^2\,dt. \tag{38}$$

By a second appeal to the theory of solution in power-series we see that $\xi(xt)$, $\bar{\xi}(Xt)$, as functions of t, will have the forms

$$P_4(t^3)+t^{3p_1+1}P_5(t^3)+t^{3p_2+2}P_6(t^3),$$
$$P_7(t^3)+t^{-(3p_1+1)}P_8(t^3)+t^{-(3p_2+2)}P_9(t^3).$$

* Cf. (6), 382 (25).

† That is to say, unless $3q_1+1$, $3q_2+2$ or their difference $3(q_1-q_2)-1$ is zero or a positive or negative integer of the form $3n$, in which case logarithmic solutions appear.

Hence $\xi(xt)\bar{\xi}(Xt)P(t^3)$ will contain no term in t^{-3} unless, exceptionally, either $3p_1+1$, $3p_2+2$ or their difference $3(p_1-p_2)-1$ is zero or a positive or negative integer of the form $3n$. Thus, in general, the integrand of (38) has zero residue at $t = 0$, and so $\sum U_\omega = 0$, which is the null condition.

In the exceptional cases in which either (or both) of the differential equations (30), (32) has a logarithmic singularity, I shall be content to suppose, by the principle of continuity, that the null condition persists: I have no doubt that a closer analysis would confirm this.

It should be observed that the foregoing argument about the null condition does not require us to suppose p_1, p_2, q_1, q_2 positive integers, and that it evidently extends to equations of any order.

8. The asymptotic values of $\xi(x)$, $\eta(y)$

To determine the behaviour of the contour-integral for U_ω along the associated characteristic it is (naturally) necessary to fix the solutions $\xi(x)$, $\eta(y)$ of the differential equations precisely, except for possible constant multipliers which can be adjusted in the final product $\xi\bar{\xi}\eta\bar{\eta}$. I shall define these solutions by means of their asymptotic values for large complex x, y. When the ξ, η have been found, we can then determine the 'associated' $\bar{\xi}$, $\bar{\eta}$ described above.

As we have already seen, when p_1, p_2 are positive integers, an appropriate solution ξ is

$$\xi(x) = \Delta_x(p_1)\Delta_x(p_2)e^x,$$

where the operators Δ are of degrees p_1, p_2 in δ and have leading coefficients unity. Thus their product can be expanded in the form

$$\Delta_x(p_1)\Delta_x(p_2) = x^{p_1+p_2}D^{p_1+p_2}+C_1\,x^{p_1+p_2-1}D^{p_1+p_2-1}+\ldots+C_{p_1+p_2},$$

where the C's are constant. Thus

$$\begin{aligned}\xi(x) &= e^x(x^{p_1+p_2}+C_1\,x^{p_1+p_2-1}+\ldots+C_{p_1+p_2})\\ &\equiv x^{p_1+p_2}e^xP_0(x^{-1}),\end{aligned}$$

where P_0 denotes a polynomial function of its argument with leading coefficient unity. Thus, if x is complex and $|x| \to \infty$,

$$\xi(x) \sim x^{p_1+p_2}e^x. \tag{39}$$

When p_1, p_2 are not both integers, the differential equation (30) is no longer soluble in finite terms, the polynomial $P_0(x^{-1})$ being replaced by an infinite series: and, what is worse, this infinite series is divergent. Fortunately, the asymptotic form (39) remains and is

sufficient to identify the precise solution. We could show this by extending to equations of higher order a method given by E. L. Ince* to determine the asymptotic form of solutions of analogous (but more general) equations of the second order. I shall be content here to notice that, if in (30) we put

$$u \equiv x^{p_1+p_2}e^x\bar{u},$$

the equation can be written

$$\left(D^3-1-\frac{b}{x^2}D-\frac{c}{x^3}\right)e^x\bar{u} = 0,$$

where b, c are constants, i.e.

$$(D^2+3D+3)D\bar{u} = \frac{1}{x^2}\left(bD+b+\frac{c}{x}\right)\bar{u}.$$

The application of Cauchy's existence-theorem to an equation of this form indicates the existence of a solution which converges to an arbitrary constant, say unity, as $|x| \to \infty$.

Accordingly, I define, in terms of asymptotic form, the solutions

$$\left.\begin{array}{lll} \xi_1 \sim x^{p_1+p_2}e^x, & \xi_2 \sim x^{p_1+p_2}e^{\omega x}, & \xi_3 \sim x^{p_1+p_2}e^{\omega^2 x} \\ \eta_1 \sim y^{q_1+q_2}e^y, & \eta_2 \sim y^{q_1+q_2}e^{\omega y}, & \eta_3 \sim y^{q_1+q_2}e^{\omega^2 y} \end{array}\right\}. \tag{40}$$

Substituting these asymptotic forms in the Wronskian of ξ_1, ξ_2, ξ_3 we at once have for the 'associated' adjoint solutions

$$\bar{\xi}_1 \sim \tfrac{1}{3}x^{-(p_1+p_2+2)}e^{-x}, \quad \bar{\xi}_2 \sim \tfrac{1}{3}\omega x^{-(p_1+p_2+2)}e^{-\omega x}, \quad \bar{\xi}_3 \sim \tfrac{1}{3}\omega^2 x^{(p_1+p_2+2)}e^{-\omega^2 x}, \tag{41}$$

with corresponding asymptotic forms for $\bar{\eta}_1$, $\bar{\eta}_2$, $\bar{\eta}_3$.†

Now take the set of fundamental solutions in the modified form‡

$$U_r = \frac{1}{2\pi i}\int^{(0+)} \xi_1(xt)\eta_r(yt)\bar{\xi}_1(Xt)\bar{\eta}_r(Yt)t^2\,dt \quad (r = 1, 2, 3), \tag{42}$$

where the characteristic $(x-X)+\omega(y-Y) = 0$ now corresponds to the suffix 2. To determine the value of U_2 along this characteristic I expand the contour of integration into the 'infinite' circle: in virtue of the asymptotic forms (40), (41) we then have

$$U_2 \sim \frac{\omega}{6\pi i}\int (xt)^{p_1+p_2}(yt)^{q_1+q_2}(Xt)^{-(p_1+p_2+2)}(Yt)^{-(q_1+q_2+2)}e^{(x+\omega y-X-\omega Y)t}t^2\,dt.$$

* (7), 169–71 (§ 7.31).

† These asymptotic forms agree with those derivable from the expressions given for $\bar{\xi}$, $\bar{\eta}$ in § 6.

‡ In particular a factor $\frac{1}{3}$ has been removed since it now appears in the asymptotic forms (41).

Define $$W_2 \equiv x^{-(p_1+p_2)}y^{-(q_1+q_2)}X^{p_1+p_2+2}Y^{q_1+q_2+2}U_2, \tag{43}$$

so that $$W_2 \sim \frac{\omega}{6\pi i}\int e^{(x+\omega y-X-\omega Y)t}\frac{dt}{t^2},$$

and therefore $$\frac{\partial W_2}{\partial x} \sim \frac{\omega}{6\pi i}\int e^{(x+\omega y-X-\omega Y)t}\frac{dt}{t}.$$

Hence, along the characteristic $x+\omega y = X+\omega Y$,

$$W_2 = 0, \qquad \frac{\partial W_2}{\partial x} = \tfrac{1}{3}\omega,$$

and so, by (43),

$$U_2 = 0, \qquad \frac{\partial U_2}{\partial x} = \tfrac{1}{3}\omega x^{p_1+p_2}y^{q_1+q_2}X^{-(p_1+p_2+2)}Y^{-(q_1+q_2+2)}.$$

These (with the similar result for the y-derivative of U_2) are the 'characteristic' conditions (C) for the equation we are considering.*

It is not difficult to see that this argument extends to the corresponding equation of any order.

9. An explicit form for the functions $\xi(x)$, etc.

Explicit expressions can be obtained for the functions ξ, η, $\bar{\xi}$, $\bar{\eta}$ in the form of $(m-1)$-fold integrals. I consider only functions of the third order and, in particular, equation (30)

$$\delta(\delta-3p_1-1)(\delta-3p_2-2)u = x^3u, \tag{30}$$

where I suppose the factors on the left arranged in descending order, i.e.

$$3p_2+2 > 3p_1+1 > 0. \tag{44}$$

In the first instance restrict p_1, p_2 to be integers and operate on the differential equation with

$$x^{-(3p_2+2)}(\delta-3)...(\delta-3p_2-3).(\delta-3p_1-4)...(\delta-3p_2-4).$$

This gives, after reduction,

$$\{\delta(\delta-1)(\delta-2)-x^3\}x^{-(3p_2+2)}\times$$
$$\times\delta...(\delta-3p_2).(\delta-3p_1-1)...(\delta-3p_2-1)u = 0,$$

where the operators in each group descend by differences of 3. Thus, for a suitably chosen solution $u = \xi$, we have

$$x^{-(3p_2+2)}\delta...(\delta-3p_2).(\delta-3p_1-1)...(\delta-3p_2-1)\xi = e^x,$$

i.e. $$\left(\frac{d}{x^2dx}\right)^{p_2+1}x^{3p_2+4}\left(\frac{d}{x^2dx}\right)^{p_2-p_1+1}x^{-(3p_1+1)}\xi = \frac{e^x}{x}. \tag{45}$$

* Cf. equation (28) above.

Now it is easily verified that a particular solution of the differential equation

$$\left(\frac{d}{x^2dx}\right)^{n+1} f(x) = g(x)$$

is given by
$$f(x) = \int_h^x g(t)\left(\frac{x^3-t^3}{3}\right)^n \frac{t^2\,dt}{n!},$$

where h is any convenient constant.

If we take $h = -\infty$ and apply this formula twice in the differential equation (45), we get at length

$$\xi(x) = x^{3p_1+1}\int_{-\infty}^{x}\left(\frac{x^3-t^3}{3}\right)^{p_2-p_1}\frac{t^{-(3p_2+2)}}{(p_2-p_1)!}\int_{-\infty}^{t} e^{\tau}\left(\frac{t^3-\tau^3}{3}\right)^{p_2}\frac{\tau d\tau dt}{p_2!}. \quad (46)$$

We suppose the real part of x negative to avoid difficulties at $t = 0$ on the path of integration in t. For more general values of x this double integral would need to be extended into a double contour-integral of the type introduced by Hankel for Bessel's functions.

Removing the integer-restriction on p_1, p_2 but retaining the inequalities (44), we can replace this double integral by

$$\xi(x) = \frac{x^{3p_1+1}}{\Gamma(p_2-p_1+1)\Gamma(p_2+1)}\int_{-\infty}^{x}\left(\frac{x^3-t^3}{3}\right)^{p_2-p_1} t^{-(3p_2+2)}\int_{-\infty}^{t} e^{\tau}\left(\frac{t^3-\tau^3}{3}\right)^{p_2}\tau d\tau dt. \quad (47)$$

So far I have proved (47) a solution of the equation (45) but not necessarily of the simpler original equation (30). To complete the argument write

$$t = x/t', \qquad \tau = x/t'\tau'$$

and drop accents. Then, after reduction and extruding a certain constant C, we have

$$\begin{aligned}\xi &= Cx^{3p_2+2}\int_0^1\int_0^1 e^{x/t\tau}t^{-1}(1-t^{-3})^{p_2-p_1}\tau^{-2}(1-\tau^{-3})^{p_2}\frac{dtd\tau}{t\tau} \\ &= Cx^{3p_2+2}\xi' \quad \text{say,} \qquad (48)\end{aligned}$$

where
$$\xi' \equiv \int_0^1\int_0^1 e^{x/t\tau}\phi(t)\psi(\tau)\frac{dtd\tau}{t\tau},$$

and
$$\phi(t) \equiv t^{-1}(1-t^{-3})^{p_2-p_1}, \qquad \psi(\tau) \equiv \tau^{-2}(1-\tau^{-3})^{p_2}.$$

We can easily verify that ϕ, ψ satisfy the respective equations

$$(\delta_t+3p_2-3p_1+1)\phi = (\delta_t-2)t^3\phi, \qquad (\delta_\tau+3p_2+2)\psi = (\delta_\tau-1)\tau^3\psi.$$

Then ξ' is a double integral of Mellin's type such that, if certain expressions vanish at the limits 1, 0,

$$\delta(\delta+3p_2-3p_1+1)(\delta+3p_2+2)\xi'$$
$$= \delta(\delta-1)(\delta-2)\int_0^1\int_0^1 e^{x/t\tau}t^3\phi(t)\tau^3\psi(\tau)\frac{dtd\tau}{t\tau} = x^3\xi'.$$

Thus, from (48), $\quad \delta(\delta-3p_1-1)(\delta-3p_2-2)\xi = x^3\xi,$

if the limits in the double integral have been well chosen.

It is perhaps easier to interpret the integral for ξ' as a double Laplace integral, writing the differential equation for ξ' as

$$\Delta\xi' \equiv \{x^2(D^3-1)+3(p+q+2)xD^2+(3p+3)(3q+2)D\}\xi' = 0, \quad (49)$$

where $p \equiv p_2$, $q \equiv p_2-p_1$. Writing also $t = 1/t'$, $\tau = 1/\tau'$ in the integral for ξ' and dropping accents we have

$$\xi' = \int_1^\infty\int_1^\infty e^{xt\tau}(1-t^3)^q(1-\tau^3)^p\tau\, dtd\tau.$$

Substituting this in (49) we obtain, on reduction,

$$\Delta\xi' = -x\int_1^\infty (1-\tau^3)^p\big[e^{xt\tau}(1-t^3)^{q+1}\big]_{t=1}^\infty d\tau -$$
$$-\int_1^\infty t(1-t^3)^q\big[e^{xt\tau}(1-\tau^3)^{p+1}(xt\tau+3q+2)\big]_{\tau=1}^\infty dt,$$

which vanishes when $x < 0$, $p+1$, $q+1 > 0$, the last two inequalities being secured by (44).

To investigate the asymptotic form write in (47)

$$\tau = t-\tau', \qquad t = x-t'$$

and drop accents. Then, after reduction,

$$\xi = \frac{x^{p_1+p_2}e^x}{\Gamma(p_2-p_1+1)\Gamma(p_2+1)}\int_0^\infty\int_0^\infty e^{-t-\tau}\left(t-\frac{t^2}{x}+\frac{t^3}{3x^2}\right)^{p_2-p_1}\times$$
$$\times\left\{\tau-\frac{\tau^2}{x-t}+\frac{\tau^3}{3(x-t)^2}\right\}^{p_2}\left(1-\frac{t}{x}\right)^{-(p_2+2)}\left(1-\frac{t+\tau}{x}\right)dtd\tau.$$

Thus, for x of large absolute value,

$$\xi \sim \frac{x^{p_1+p_2}e^x}{\Gamma(p_2-p_1+1)\Gamma(p_2+1)}\int_0^\infty e^{-t}t^{p_2-p_1}\,dt\int_0^\infty e^{-\tau}\tau^{p_2}\,d\tau = x^{p_1+p_2}e^x.$$

By (41) this form identifies ξ as the required ξ_1. We can evidently write down analogous double integrals having the asymptotic form of ξ_2, ξ_3 and therefore to be identified with them.* Further, we can write down similar double integrals which are solutions of the differential equation for $\bar{\xi}$ and have the same asymptotic forms as $\bar{\xi}_1$, $\bar{\xi}_2$, $\bar{\xi}_3$; these, then, must be the same as the 'associated' $\bar{\xi}_1$, $\bar{\xi}_2$, $\bar{\xi}_3$ as calculated from the Wronskian of ξ_1, ξ_2, ξ_3. We deal similarly with η, $\bar{\eta}$.

10. The second-order equation

I conclude with a brief account of the equation of the second order for which the theory is somewhat simpler. In the first place, the *set* of fundamental solutions reduces to the single fundamental solution of Riemann's theory: more precisely, there is a pair of such solutions, which, in virtue of the null condition, are equal and opposite, and, naturally, it is sufficient to pick out one of these and regard it as the only fundamental solution $U(X,Y;x,y)$.

I shall take the partial differential equation and its adjoint to be

$$y^2\delta(\delta+2p+1)V = x^2\delta'(\delta'+2q+1)V, \tag{50}$$

$$y^2\delta(\delta-2p-1)U = x^2\delta'(\delta'-2q-1)U. \tag{51}$$

The characteristics are the lines $x\pm y =$ constant, and the characteristic condition (C) now requires that, along each of the two characteristics $x-X = \pm(y-Y)$,

$$U = x^p y^q X^{-(p+1)} Y^{-(q+1)}. \tag{52}$$

As before, when p, q are positive integers, we can give the fundamental solution in the form

$$U = X^{-(2p+1)}Y^{-(2q+1)}\Delta_x(p)\Delta_y(q)\Delta_X(p)\Delta_Y(q)\frac{\{(x-X)\pm(y-Y)\}^{2p+2q}}{(2p+2q)!}, \tag{53}$$

where

$$\Delta_x(p) \equiv (\delta-1)(\delta-3)...(\delta-2p+1),$$

$$\Delta_y(q) \equiv (\delta'-1)(\delta'-3)...(\delta'-2q+1),$$

and $\Delta_X(p)$, $\Delta_Y(q)$ are similar operators in X, Y. In the operand

* For no $C_1\xi_1+C_2\xi_2+C_3\xi_3$ has, for all x, the same asymptotic form as an ξ_r except, of course, ξ_r itself.

of (53) the choice of either sign in $\pm$ leads to the same expression for U: this may be regarded as an aspect of the null condition.

In expressing U as a contour-integral the differential equations defining ξ, η are

$$\delta(\delta-2p-1)\xi = x^2\xi, \qquad \delta'(\delta'-2q-1)\eta = y^2\eta, \tag{54}$$

which are evidently variants of Bessel's equation. The first of these equations has the solutions

$$x^{p+\frac{1}{2}}K_{p+\frac{1}{2}}(\pm x).$$

More exactly, we remark that, when p is a positive integer,*

$$(\delta-1)(\delta-3)...(\delta-2p+1)e^{-x} = (-)^p\sqrt{\left(\frac{2}{\pi}\right)}x^{p+\frac{1}{2}}K_{p+\frac{1}{2}}(x).$$

Replacing $p+\frac{1}{2}$, $q+\frac{1}{2}$, for convenience, by μ, ν, we can define

$$U = \frac{2i}{\pi^3}\left(\frac{x}{X}\right)^\mu\left(\frac{y}{Y}\right)^\nu \int^{(0+)} K_\mu(-xt)K_\nu(-yt)K_\mu(Xt)K_\nu(Yt)t\,dt. \tag{55}$$

Changing signs in the argument of K_ν, consider alternatively

$$U' \equiv \frac{2i}{\pi^3}\left(\frac{x}{X}\right)^\mu\left(\frac{y}{Y}\right)^\nu \int^{(0+)} K_\mu(-xt)K_\nu(yt)K_\mu(Xt)K_\nu(-Yt)t\,dt. \tag{56}$$

Then $U'-U$

$$= \frac{2i}{\pi^3}\left(\frac{x}{X}\right)^\mu\left(\frac{y}{Y}\right)^\nu \int^{(0+)} K_\mu(-xt)K_\mu(Xt)\{K_\nu(yt)K_\nu(-Yt)-K_\nu(-yt)K_\nu(Yt)\}t\,dt. \tag{57}$$

Now on substituting first† for $K_\nu(-Yt)$, $K_\nu(-yt)$ and then‡ for $K_\nu(yt)$, $K_\nu(Yt)$, we reduce

$$K_\nu(yt)K_\nu(-Yt)-K_\nu(-yt)K_\nu(Yt)$$

to

$$\tfrac{1}{2}i\pi^2 \operatorname{cosec}\nu\pi\{I_\nu(yt)I_{-\nu}(Yt)-I_\nu(Yt)I_{-\nu}(yt)\},$$

and this can be expanded in a power-series $P(t^2)$ containing only non-negative integer powers of t^2, since each term can be separately so expanded.§ $K_\mu(-xt)$, $K_\mu(Xt)$ can each be expanded in power-series of the form $t^\mu P_1(t^2)+t^{-\mu}P_2(t^2)$, and so, as in the argument of §7, the integrand of (57), in general, has zero residue at $t = 0$, and accordingly $U = U'$. This equivalence, in the second-order equation,

* Cf. (8), 79 (6) and 80 (13).

† From (8), 80 (18).

‡ From (8), 78 (6).

§ Cf. (8), 77 (2).

of the two forms (55), (56) for the fundamental solution, corresponds, of course, to the null condition in equations of higher order.

Expanding the contour in (55) into the 'infinite' circle and so replacing the Bessel's functions by their asymptotic forms,* we obtain (returning from μ, ν to $p+\frac{1}{2}$, $q+\frac{1}{2}$)

$$U \sim \frac{1}{2\pi i} x^p y^q X^{-(p+1)} Y^{-(q+1)} \int e^{(x-X+y-Y)t} \frac{dt}{t}. \tag{58}$$

Thus, along the characteristic $x+y = X+Y$,

$$U = x^p y^q X^{-(p+1)} Y^{-(q+1)}; \tag{59}$$

similarly U' has the same value along the other characteristic $x-y = X-Y$. In view of the equivalence of U, U' this establishes the 'characteristic' condition (C).†

For purposes of reference it is perhaps better to use a substitution of the type (17 A), (17 B) to transform the equation (51) into a 'normal form'

$$y^2(\delta^2-\mu^2)U_1 = x^2(\delta'^2-\nu^2)U_1 \tag{60}$$

with the fundamental solution

$$U_1 = \frac{2i}{\pi^3} \int^{(0+)} K_\mu(-xt) K_\nu(\mp yt) K_\mu(Xt) K_\nu(\pm Yt) t\, dt. \tag{61}$$

By a substitution of the same type we can transform the equation into

$$y^2(\delta+p)(\delta-p-1)U_2 = x^2(\delta'+q)(\delta'-q-1)U_2,$$

i.e.

$$\frac{\partial^2 U_2}{\partial x^2} - \frac{\partial^2 U_2}{\partial y^2} = \left\{\frac{p(p+1)}{x^2} - \frac{q(q+1)}{y^2}\right\} U_2. \tag{62}$$

The change of independent variables

$$x = x_1+y_1, \qquad y = x_1-y_1$$

then gives

$$\frac{\partial^2 U_2}{\partial x_1\, \partial y_1} = \left\{\frac{p(p+1)}{(x_1+y_1)^2} - \frac{q(q+1)}{(x_1-y_1)^2}\right\} U_2. \tag{63}$$

* From (8), 202, § 7.23 (1).

† Reference to the condition as formulated in § 2 will suggest that a factor $\frac{1}{2}$ has been omitted in (59). Actually this is accounted for by a slight difference between the technique that I use for the equation with m characteristics and Riemann's method applicable to the equation with two characteristics. To reconcile them we could here have taken the fundamental solution to be $\frac{1}{2}(U+U')$.

Now this is a case of Darboux's equation which I have discussed in (**5**), giving its fundamental solution (in the notation of that article) as

$$U_2 = \Lambda_2\begin{bmatrix} p & q \\ \xi & \eta \end{bmatrix},$$

where $$\xi \equiv -\frac{(x_1-X_1)(y_1-Y_1)}{(x_1+y_1)(X_1+Y_1)}, \qquad \eta \equiv \frac{(x_1-X_1)(y_1-Y_1)}{(x_1-y_1)(X_1-Y_1)}.$$

Since the fundamental solution of a partial differential equation of the form $s = \lambda z$ is known to be unique, we can identify this expression for U_2 with that obtained from (55), thus getting (in the more convenient variables x, y)*

$$\frac{2i}{\pi^3}\sqrt{(xyXY)}\int^{(0+)} K_{p+\frac{1}{2}}(-xt)K_{q+\frac{1}{2}}(-yt)K_{p+\frac{1}{2}}(Xt)K_{q+\frac{1}{2}}(Yt)t\,dt = \Lambda_2\begin{bmatrix} p & q \\ \xi & \eta \end{bmatrix}, \tag{64}$$

where $$\xi = \frac{(x-X)^2-(y-Y)^2}{4xX}, \qquad \eta = \frac{(x-X)^2-(y-Y)^2}{4yY}. \tag{65}$$

By using one of the expressions for Λ_2 given in (**5**), for example†

$$\Lambda_2\begin{bmatrix} p & q \\ \xi & \eta \end{bmatrix} = \left(\xi\frac{\partial}{\partial\xi}+\eta\frac{\partial}{\partial\eta}+1\right)\int_0^1 P_p(1-2\xi+2\xi t)P_q(1-2\eta t)\,dt$$

we express (64) as an identity connecting Bessel's and Legendre's function (which I have not so far found in the literature).

* In transforming from U to U_2 we have also to make an adjustment for the change from δ, δ' to $\partial/\partial x$, $\partial/\partial y$ as the operators defining 'adjoint'. The final form can be checked by noticing that the equations (62), (63) are self-adjoint and that accordingly U_2 should be symmetrical in (x, y) and (X, Y).

† This is (**5**), 237 (9).

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